Curves & Surfaces

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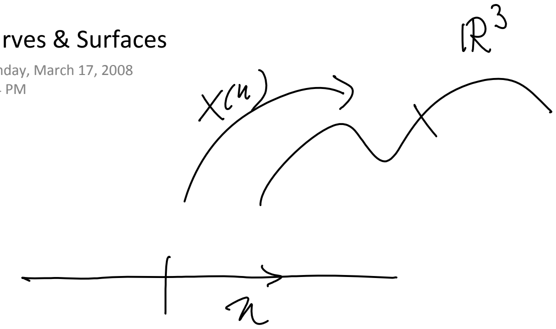

$$\begin{cases} \text{curvature} & \kappa(u) \geq 0 \\ \text{torsion} & \tau(u) \end{cases}$$

invariant to rigid body transformations

$$R = [T\ N\ B] \in SO(3) \qquad R^t R = I \quad |R| = 1$$

$$\begin{cases} \dot{T} = \kappa N \\ \dot{N} = -\kappa T + \tau B \\ \dot{B} = -\tau N \end{cases}$$

$$\dot{R} = [\dot{T}\ \dot{N}\ \dot{B}] = \underbrace{[T\ N\ B]}_{R} \underbrace{\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}}_{K}$$

K skew symmetric

$$\dot{R} = RK$$

problem: given functions $\kappa(u) \geq 0$ and $\tau(u)$, is there a curve $x(u)$ so that $\kappa = \text{curvature}(x)$ and $\tau = \text{torsion}(x)$? What about uniqueness?

Rotations

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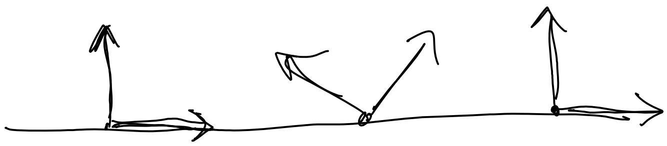
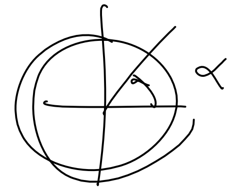




$t=0$ $\qquad$ $t=1$

$$\begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\end{pmatrix}$$

$\alpha = 0$ $\qquad$ $\alpha = 2\pi$

$$\cancel{R_\alpha}\; e^{\alpha J} = \cos(\alpha)\, I + \sin(\alpha)\, J = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}$$

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha) \qquad j = \sqrt{-1}$$

$$e^{J\alpha} \qquad J^0 = I \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$J^3 = -J$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \qquad e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

$$J^n = \begin{cases} (-1)^{n/2} I \\ J(-1)^{\frac{n-1}{2}} \end{cases}$$

$$e^M e^N = e^{M+N}$$

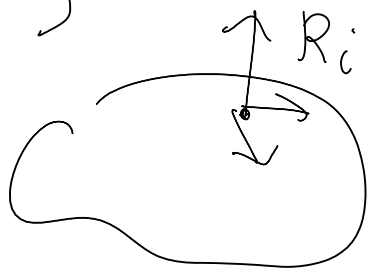
W skew symmetric $\Rightarrow$ e^W orthogonal

R

$$W = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ \end{bmatrix} =$$

$$W = \begin{bmatrix} -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix} =$$

$$R_i = e^{W_i}$$



Skew-symmetric matrices

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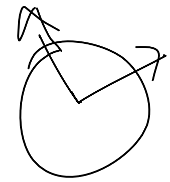
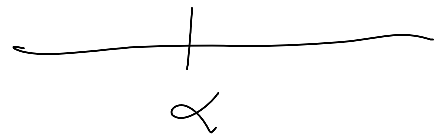
$$\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} = \alpha J \qquad e^{\alpha J}$$





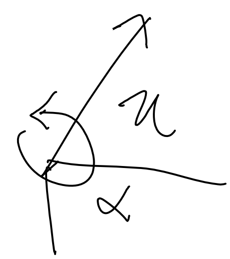
$$W = \underbrace{\begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & w_1 & 0 \end{pmatrix}}_{\omega} \leftarrow \quad w \times x = W x$$

$$R = e^{\hat{\omega}} = e^{\alpha \hat{u}}$$

$\alpha = \|\omega\|$

$\|u\| = 1$

$$e^{(\alpha + 2\pi)\hat{u}} = e^{\alpha \hat{u}}$$



Laplacian coordinates

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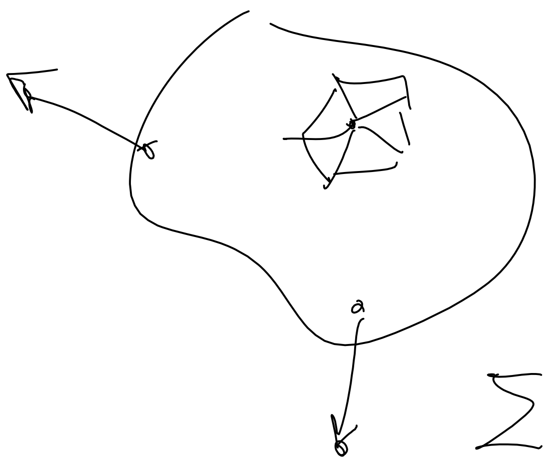

$$\Delta x_i = \sum_{j \in i^*} w_{ij} (x_j - x_i)$$

$$\delta_i = \Delta x_i^0$$

$$\sum_i \mu_i \| \Delta x_i - \delta_i \|^2 = E(x)$$

$$x_j = x_j^1 \quad j \in I$$

$$\sum \mu_i \| \Delta x_i - R_i \delta_i \|^2$$

I

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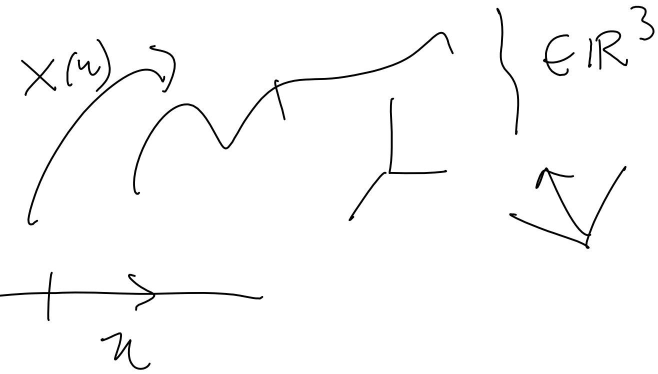

$$\begin{cases} \kappa(u) & \text{curvature} \geq 0 \\ \tau(u) & \text{torsion} \end{cases}$$

$$T = \frac{\dot{X}}{\|\dot{X}\|} \qquad \kappa = \|\dot{T}\| \geq 0 \qquad \dot{T} = \kappa N$$

$$B = T \times N$$

$$R = \begin{bmatrix} T & N & B \end{bmatrix} \in SO(3) \qquad R^t R = I, \ |R| = 1$$

$$\begin{cases} \dot{T} = & & \kappa N & \\ \dot{N} = & -\kappa T & & + \tau B \\ \dot{B} = & & -\tau N & \end{cases}$$

$$\dot{R} = \underbrace{\begin{bmatrix} T & N & B \end{bmatrix}}_{R} \underbrace{\begin{pmatrix} 0 & -\kappa & \\ \kappa & 0 & -\tau \\ 0 & \tau & \end{pmatrix}}_{K}$$

$$\underline{\dot{R} = R K}$$ ordinary differential equation in u

Surfaces

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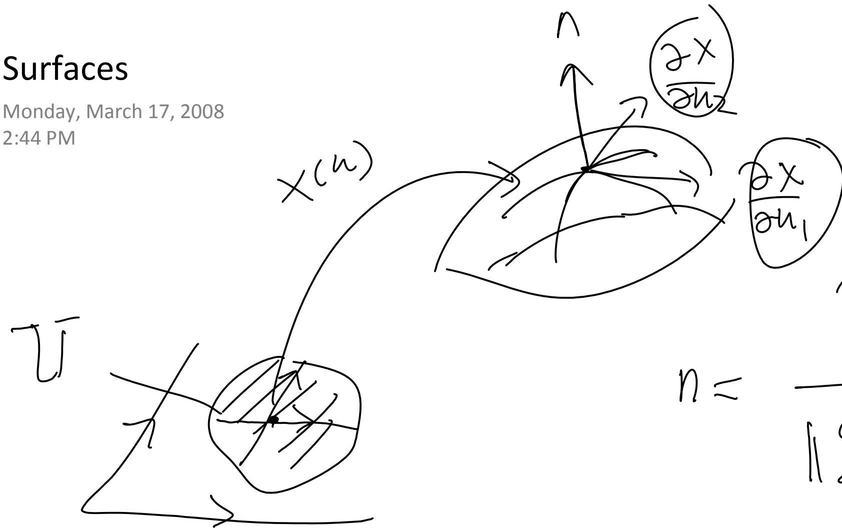
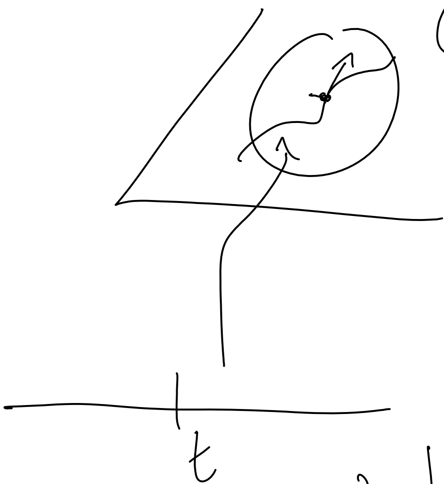


$$\begin{bmatrix}\frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & n\end{bmatrix}$$

$$n = \frac{\frac{\partial x}{\partial u_1}\times\frac{\partial x}{\partial u_2}}{\left\|\frac{\partial x}{\partial u_1}\times\frac{\partial x}{\partial u_2}\right\|}$$

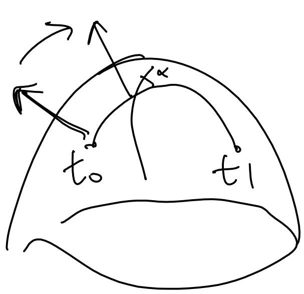


$x(u(t))$

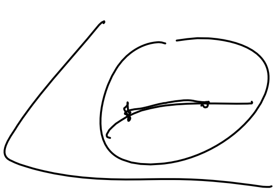
$$\frac{d}{dt}\{x(u(t))\} = \frac{\partial x}{\partial u_1}(u(t))\cdot\underbrace{\frac{du_1}{dt}}_{\in\mathbb{R}} + \frac{\partial x}{\partial u_2}(u(t))\underbrace{\frac{du_2}{dt}}_{\in\mathbb{R}}$$

$$3\,\Big|\,\overset{2}{\overline{Dx}} = \begin{pmatrix}\frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2}\end{pmatrix}$$

$$\frac{d}{dt}\left(x(u(t))\right) = \underline{Dx(u)\cdot u'}$$



length $= \int_{t_0}^{t_1}\left\|\frac{d}{dt}\{x(u(t))\}\right\|\,dt$



$$= \int_{t_0}^{t_1}\sqrt{(u')^t\,G\,u'}\;dt$$

$$G = Dx^t Dx \quad - \quad 2 \times 2$$

$$\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{bmatrix} \cdot & \\ & \cdot \end{bmatrix}$$