Integration of Curves from Frenet Frame and speed function

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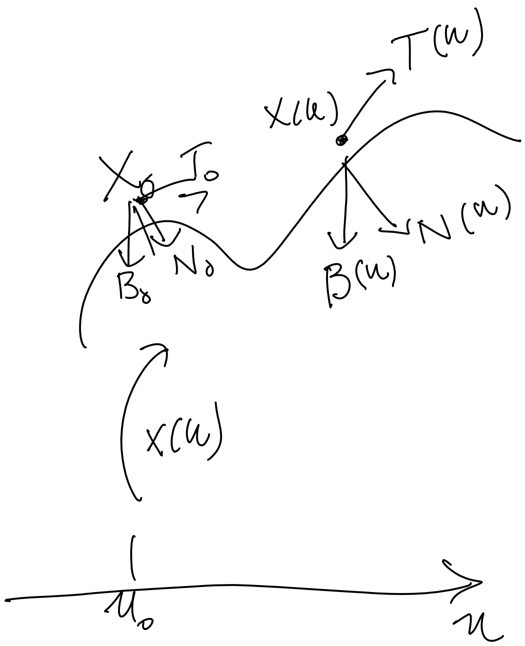


$$R = [T\ N\ B] \in SO(3)$$

Frenet Frame forms an orthonormal frame for each point along the curve

$$\underbrace{[T\ N\ B]'}_{R'(u)} = \underbrace{[T\ N\ B]}_{R(u)} \underbrace{\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}}_{K(u)}$$

Skew-symmetric matrix $K(u)$

Speed function

Ordinary differential equation has a unique solution

$$\begin{cases} X'(u) = d(u)\, T(u) \\ X(u_0) = X_0 \end{cases}$$

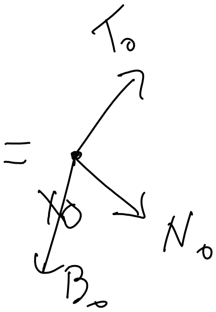
Given this data $\begin{cases} d(u) = \|X'(u)\| > 0 \\ T(u), \quad \|T(u)\| = 1 \end{cases}$

Note that the curve is parameterized by arc-length if and only if d(u)=1 for all u.

But we don't have the tangent vector field. What we have is curvature, torsion, and the initial frame.
To get the tangent vector field we need to solve the following matrix ordinary differential equation.

$$\begin{cases} R'(u) = R(u)\, K(u) \\ R(u_0) = R_0 \end{cases}$$

$R_0 =$



For an arbitrary skew-symmetric matrix field K(u), a unique solution exists,
However, we need to prove that the solution R(u) is orthogonal for all u.
So far, we only know this for only one value of u.

A 3x3 skew-symmetric matrix has the following form

$$K = \hat{k} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

where

$$k = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \qquad \text{and} \qquad K + K^t = 0$$

For any vector v, we have $Kv = \hat{k} v = k \times v$

In particular $\hat{k}\, k = k \times k = 0$

Looking at the differential equation

$$R' = RK \quad \Rightarrow R^t R' = K$$

$$0 = K + K^t = R^t R' + (R')^t R = \frac{d}{du}\left\{ R^t R \right\}$$

i.e., $R^t R$ is constant, then $\forall u$:

$$R(u)^t R(u) = R(u_0)^t R(u_0) = R_0^t R_0 = I$$

Matrix K continuous case

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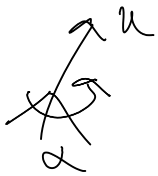
intuitive interpretation

$R' = RK \qquad K = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$

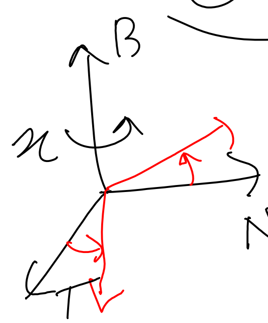


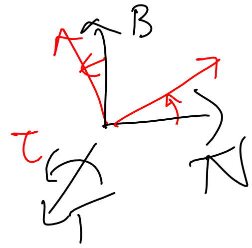
$$e^{\hat{\omega}} = Rot(u, \alpha) \qquad K = \underbrace{\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 \\ 0 \\ \kappa \end{bmatrix}^\wedge} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}}_{\begin{bmatrix} \tau \\ 0 \\ 0 \end{bmatrix}^\wedge}$$

where $\omega = \alpha u$

$|u| = 1 \qquad \alpha > 0$

$$e^{\begin{bmatrix} 0 \\ 0 \\ \kappa \end{bmatrix}^\wedge} \qquad e^{\begin{bmatrix} \tau \\ 0 \\ 0 \end{bmatrix}^\wedge}$$





in general ?

$$e^{\begin{bmatrix} 0 \\ 0 \\ \kappa \end{bmatrix}^\wedge} e^{\begin{bmatrix} \tau \\ 0 \\ 0 \end{bmatrix}^\wedge} \neq e^{\begin{bmatrix} \tau \\ 0 \\ \kappa \end{bmatrix}^\wedge}$$

because

$e^{A+B} = e^{A} e^{B}$ only if $AB = BA$

However,

Matrix K Discrete case

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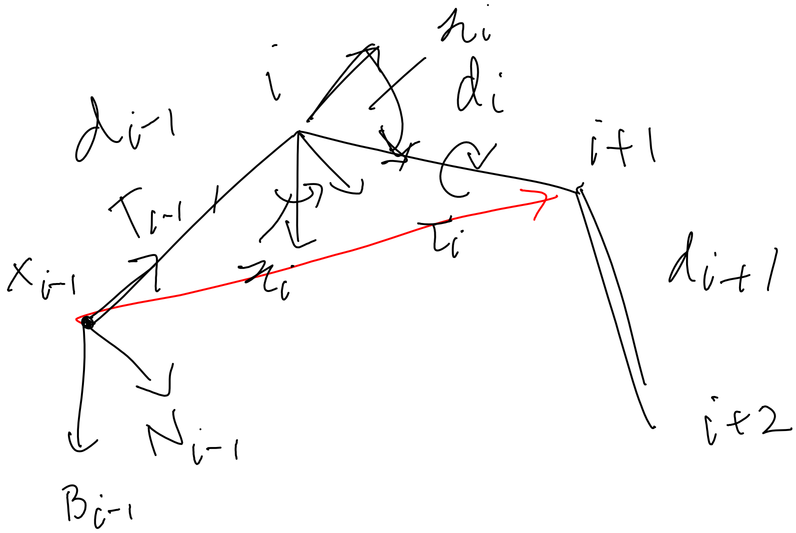


$$d_{i-1} = \|x_i - x_{i-1}\|$$

$$R_{i-1} \begin{cases} T_{i-1} = \dfrac{x_i - x_{i-1}}{d_{i-1}} \\[2ex] N_{i-1} = \dfrac{(I - T_{i-1}T_{i-1}^t)(x_{i+1} - x_{i-1})}{\|(I - T_{i-1}T_{i-1}^t)(x_{i+1} - x_{i-1})\|} \\[2ex] B_{i-1} = T_{i-1} \times N_{i-1} \end{cases}$$

Integration:

$$\text{Given} \begin{cases} x_0,\ R_0,\ d_1, \ldots, d_{N-1} \\ \kappa_1, \ldots, \kappa_{N-1}, \\ \tau_1, \ldots, \tau_{N-2} \end{cases}$$

compute $x_1, \ldots, x_N$

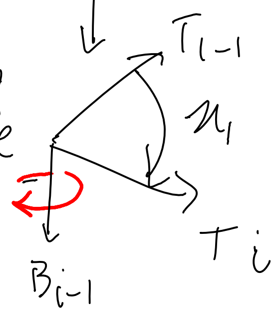
, for $i = 1, \ldots, N$

$$\left\{\begin{aligned} x_i &= x_{i-1} + d_{i-1} T_{i-1} \\ R_i &= \underbrace{e^{\widehat{\begin{bmatrix}\tau_i\\0\\0\end{bmatrix}}}}_{②}\ \underbrace{e^{\widehat{\begin{bmatrix}0\\0\\\kappa_i\end{bmatrix}}}}_{①} R_{i-1} \end{aligned}\right.$$

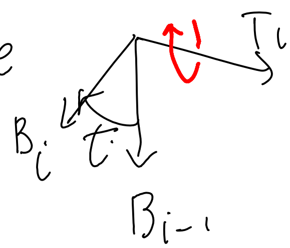
for $i = 1, \dots, N-2$

for $i = N-1$ only compute T_{N-1} from κ_{N-1} and R_{N-2}

① rotate around third coordinate vector, angle κ_i



② rotate around first coordinate vector, angle τ_i



Variational Approach

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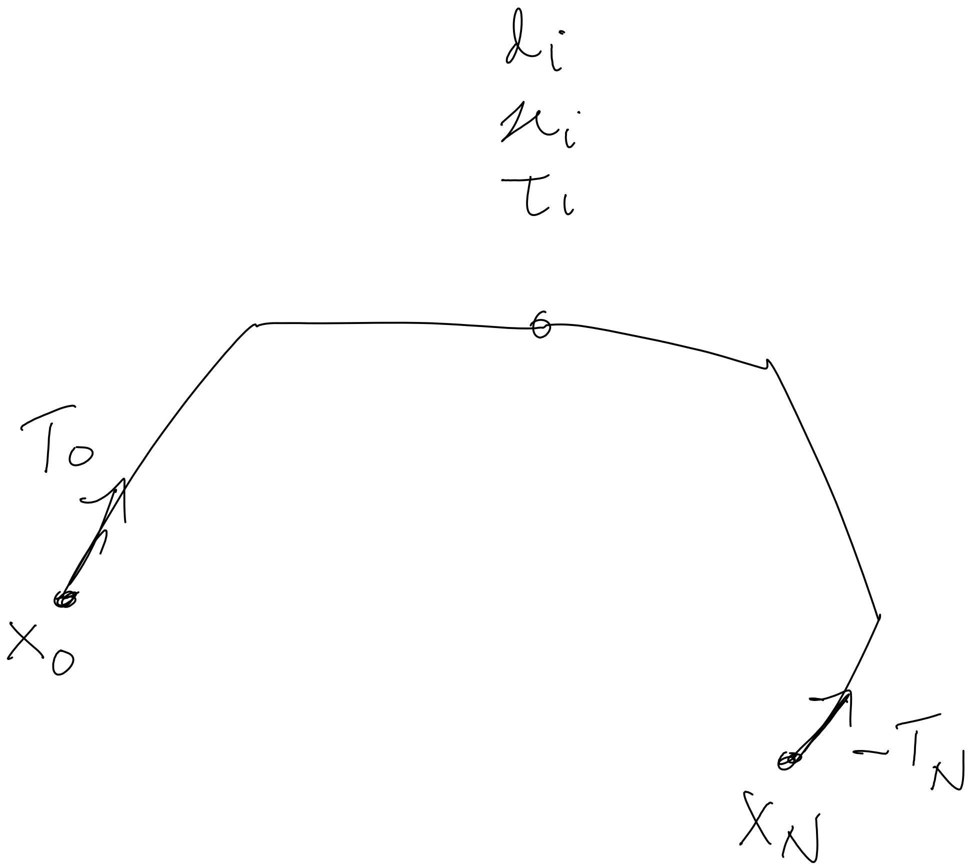

Matrix Exponential & Rotations

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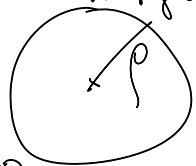
Exponential function

radius of convergence



$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$\rho = \infty$

$0 \leq \rho \leq +\infty$

$|t| < \rho$

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \qquad \sum_{n=0}^{\infty} |a_n|\,|t|^n$$

definition of radius of convergence $\quad \dfrac{1}{\rho} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$

Matrix exponential (A square matrix)

$$e^A = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} \frac{A^n}{n!}}_{} = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$\left\| \sum_{n=n_0}^{\infty} \frac{A^n}{n!} \right\|_F \leq \sum_{n=n_0}^{\infty} \frac{\|A\|_F^n}{n!}$$

converges for every matrix A

$$\|A\|_F^2 = \operatorname{trace}\{A^t A\} = \sum_{ij} a_{ij}^2$$

If W skew-symmetric $\Rightarrow e^W$ is a rotation

$$W = \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad (AB)^t = B^t A^t$$

$$\Rightarrow (W^n)^t = (W^t)^n$$

$$\left(e^{W}\right)^{t} = \left(\sum_{n=0}^{\infty} \frac{W^n}{n!}\right)^{t} = \sum_{n=0}^{\infty} \frac{(W^t)^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-W)^n}{n!} = e^{-W} \Rightarrow$$

$$W^t = -W$$

$$e^A e^B = e^{A+B}$$

$$\Uparrow$$

$$AB = BA$$

$$\left(e^{W}\right)^{t} e^{W} = e^{-W} \cdot e^{W} = e^{-W+W} = e^{0} = I$$

Rotations

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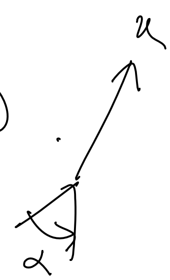
$$e^{\hat{\omega}} \quad \text{orthogonal} \quad \begin{cases} \omega = \alpha u \\ \|u\| = 1 \\ \alpha = \|\omega\| \end{cases}.$$

$\|$

$$e^{\alpha \hat{u}} \overset{?}{=} \mathrm{Rot}(u, \alpha)$$



① first The algebra

$$e^{\alpha \hat{u}} = \sum_{n=0}^{\infty} \frac{\alpha^n \hat{u}^n}{n!} = I + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \hat{u}^n = \boxed{A}$$

$$\|u\|^2 = 1$$

$$\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} = \hat{u} \iff \hat{u} x = u \times x$$

$$\hat{u}^2 = \begin{bmatrix} -u_3^2 - u_2^2 & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & -u_3^2 - u_1^2 & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & -u_2^2 - u_1^2 \end{bmatrix} =$$

$$\hat{u}^2 = u u^t - I$$

$u \times u = 0$

$$\hat{u}^3 = \hat{u}(u u^t - I) = \hat{u} u u^t - \hat{u} = -\hat{u}$$

$$\hat{u}^4 = \hat{u}(-\hat{u}) = I - u u^t$$

In general

$$\begin{cases} u^{2k} = \begin{cases} I & : k = 0 \\ (-1) \quad (I - u u^t) & : k > 0 \end{cases} \\ u^{2k+1} = (-1) \quad \hat{u} \end{cases}$$

$\wedge$ $\boxed{A}$.

$(\hat{u})^{\,n} = (-1)\ \hat{u}$

$$e^{\alpha \hat{u}} = \boxed{A}$$

$$I + \sum_{k=1}^{\infty} \frac{\alpha^{2k} (\hat{u})^{2k}}{2k!} + \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!} (\hat{u})^{2k+1}$$

$$= I + \underbrace{\left(\sum_{k=1}^{\infty} \frac{\alpha^{2k} (-1)^{k}}{(2k)!} \right)}_{(\cos(\alpha) - 1)} (I - uu^t) + \underbrace{\left[\sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!} (-1)^k \right]}_{\sin(\alpha)} \hat{u}$$

$e^{j\alpha} = \cos(\alpha) + j \sin(\alpha)$ Euler Formula, $j = \sqrt{-1}$

$$= \sum_{n=0}^{\infty} \frac{(j\alpha)^n}{n!} = \sum_{k=0}^{\infty} \frac{(j\alpha)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(j\alpha)^{2k+1}}{(2k+1)!} =$$

$$= \underbrace{\left[\sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k}}{(2k)!} \right]}_{\cos(\alpha)} + j \underbrace{\left[\sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1}}{(2k+1)!} \right]}_{\sin(\alpha)}$$

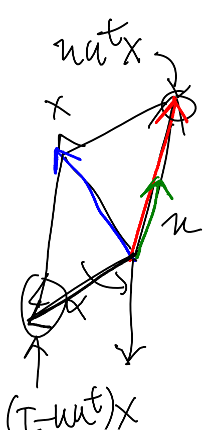
$$e^{\alpha \hat{u}} = I + \sin(\alpha)\, \hat{u} + (\cos(\alpha) - 1)(I - uu^t)$$

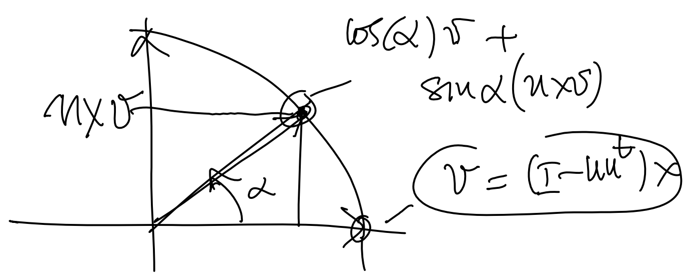
Rodrigues Formula

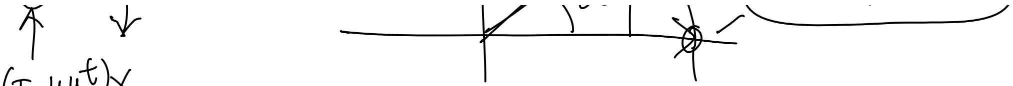
② Geometric approach

$$x = u(u^t x) + \underbrace{(I - uu^t) x}_{v}$$

$\left(u \times v = u \times x \quad \text{because} \quad u \times (u u^t x) = 0 \right)$







$(I-uu^t)x$

$\hat{n}$

$$Rot(u,\alpha)\,x = uu^t x + \sin\alpha\,(n \times x) + \cos(\alpha)\,(I-uu^t)\,x$$

$Rot(u,\alpha)\,v$

$$= \left(uu^t + \sin\alpha\,\hat{n} + \cos(\alpha)\,(I-uu^t)\right)x$$

$$\left[I + \sin\alpha\,\hat{n} + (\cos\alpha - 1)\,(I-uu^t)\right]x$$