Typical Surface Reconstruction Pipeline

- Oriented Points
- Reconstruction Method
- Surface Representation

- Positions & Normals → Watertight Surface → Polygon Mesh
Surface Representations

- Surfaces in Mathematics
  - Parametric \( S = \{ p = x(u) : u = (u_1, u_2) \in \mathbb{R}^2 \} \)
  - Implicit \( S = \{ p : f(p) = 0 \} \quad f : V \to \mathbb{R} \quad V \subset \mathbb{R}^3 \) (level set)

- We can only operate on a surface representation
  - A data structure defined by a finite number of parameters
  - Efficient to perform certain geometric operations

- Point clouds (surfaces represented as sets of samples)
  - Positions
  - Optional properties: normals, colors, etc

- Polygon meshes (piecewise planar surfaces)
  - vertices, edges, and faces
  - Optional properties: normals, color, texture coordinates, etc.

Connectivity / Watertight Surfaces

- Most applications require **connectivity information**
  - Efficient ways to find points in close proximity to each other
- Point clouds do not provide connectivity information
  - Additional data structures are needed to efficiently find neighboring points
- Connectivity is explicit in polygon meshes: **edges**
- **Triangulate** the point cloud to get connectivity information
  - Find an interpolating or approximating triangle mesh
- Many applications require **watertight** surfaces: continuous closed surfaces which partition 3D space into an **inside** and an **outside**
  - Point clouds are not watertight
  - Polygon meshes may be watertight
- Will the triangulation constructed from the point cloud be watertight?
Every regular **Implicit surface** is watertight \[ S = \{ p : f(p) = 0 \} \]

An **Isosurface** is a polygonal approximation of an implicit function associated with a volumetric grid

- Algorithm to compute an isosurface from an implicit function evaluated on the vertices of a regular hexahedral grid
- Marching Cubes: A high resolution 3D surface reconstruction algorithm
- Similar simple algorithms exist to generate isosurfaces from an implicit function evaluated on the vertices of a tetrahedral grid
- We will only discuss here approximation algorithms to fit implicit surfaces to point clouds

Related algorithms related to the Poisson Equation:


References:


Curve Reconstruction from Point Clouds

- Oriented points
- Regular grid
- Implicit function
- Isocurve
- Grid too coarse: Aliasing
- Finer grid resolves topology

IsoCurves

- Given a continuous function
  \[ f(x_1, x_2) \]
- Sampled on a regular grid
  \[ G = (V, E, C) \]
  \[ F = \{ f_v : v \in V \} \]
- Compute a polygonal approximation of a level set
  \[ C_\lambda = \{ x : f(x) = \lambda \} \]
- Increase grid resolution if necessary
The Marching Lines Algorithm (ML)

4 STEPS
1. Determine grid vertex sign bits
2. Determine supporting grid edges
3. Compute location of isovertices along supporting grid edges
4. Interconnect isovertices by table look-up within each cell

The Marching Lines Table

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Choices for 6 & 9
The 4 Steps

1. Determine grid vertex sign bits
   \[ b_v = \begin{cases} 1 & f_v \geq 0 \\ 0 & f_v < 0 \end{cases} \]

2. Determine supporting grid edges
   \[ b_v \neq b_w \]

3. Compute location of IsoVertices along supporting grid edges
   \[ \lambda = \frac{f_w}{f_w - f_v} \]

4. Interconnect IsoVertices by table look-up within each cell

Related Papers & Projects

- Vector Field Isosurface-Based Reconstruction From Oriented Points, by Sibley & Taubin, Siggraph 2005 (Sketch).
- High Resolution Surface Reconstruction from Multi-view Aerial Imagery, Calakli, Ulusoy, Restrepo, Mundy & Taubin, 3DIMPVT 2012
- REVEAL Digital Archaeology Project
- Cuneiform Automatic Translation Project
Implicit function framework

Find a scalar valued function $f : D \rightarrow \mathbb{R}$, whose zero level set $Z(f) = S'$ is the estimate for true surface $S$.

Implicit Curve and Surface Reconstruction

- **Input:** oriented point set:
  \[ D = \{ (p_i, n_i) | i=1,...,N \} \]
  contained in a bounding volume $V$

- **Output:** implicit surface
  \[ S = \{ x | f(x) = 0 \} \]
  with the function defined on $V$, such that
  \[ f(p_i) = 0 \quad \text{and} \quad \nabla f(p_i) = n_i \quad \nabla (p_i, n_i) \in D \]

- A family of implicit functions with a finite number of parameters has to be chosen

- Parameters must be estimated so that the conditions stated above are satisfied, if not exactly, then in the least-squares sense
Challenges

General Approaches

- Interpolating polygon meshes
  Boissonnat [1984], Edelsbrunner [1984]
  Amenta et al. [1998], Bernardini et al. [1999]
  Dey et al. [2003][2007], ...

- Implicit function fitting
  Taubin [1991], Hoppe et al. [1992], Curless et al. [1996]
  Whitaker [1998], Carr et al.[2001], Davis et al. [2002],
  Ohtake et al. [2004], Turk et al. [2004], Shen et al. [2004]
  Sibley-Taubin [2005]
Poisson Surface Reconstruction

1. Extend oriented points to continuous vector field $\mathbf{v}(p)$ defined on the whole volume, so that
   \[ \mathbf{v}(\mathbf{p}_i) \approx \mathbf{n}_i \]

2. Integrate vector field, by minimizing
   \[ \int_V \left\| \nabla f(p) - \mathbf{v(p)} \right\|^2 dp \]

3. Determine isolevel, by minimizing
   \[ \sum_{i=1}^{N} (f(\mathbf{p}_i) - f_0)^2 \]
What kind of implicit function?

Indicator Function

Smooth Signed Distance Function

http://mesh.brown.edu/ssd
Particularly Good at Extrapolating Missing Data

SSD Continuous Formulation

- Oriented point set:
  \[ D = \{ (p_i, n_i) \} \] sampled from a surface \( S \)

- Implicit surface:
  \[ S = \{ \mathbf{x} \mid f(\mathbf{x}) = 0 \} \] such that
  \[ f(p_i) = 0 \quad \text{and} \quad \nabla f(p_i) = n_i \quad \forall (p_i, n_i) \in D \]

- Least squares energy:
  \[
  E(f) = \sum_{i=1}^{N} f(p_i)^2 + \lambda_1 \sum_{i=1}^{N} \| \nabla f(p_i) - n_i \|^2 + \lambda_2 \int_{\Omega} \| Hf(\mathbf{x}) \|^2 \, d\mathbf{x}
  \]
What does the regularization term do?

\[
\frac{\lambda_0}{N} \sum_{i=1}^{N} f(p_i)^2 + \frac{\lambda_1}{N} \sum_{i=1}^{N} \| \nabla f(p_i) - n_i \|_2^2 + \frac{\lambda_2}{V} \int_V \| Hf(x) \|_2^2 \, dx
\]

\[Hf(x) = \begin{bmatrix}
\frac{\partial \nabla f(x)}{\partial x_1} & \frac{\partial \nabla f(x)}{\partial x_2} & \frac{\partial \nabla f(x)}{\partial x_3}
\end{bmatrix}\]

- Near data points: since the data terms dominate, the function approximates the signed distance
- Away from data points: the regularization term dominates and forces the gradient to be smooth and close to constant
Marching Cubes on Octrees

- Non-conforming hexahedral mesh
- Results in crack problem.
- Problem solved by Dual Marching Cubes

Hexahedral Envelope Using Cells of the Same Size
Non-Convex Surfaces

- Surface reconstruction from oriented points
- Uniform sampling without noise
- Interpolatory implicit surfaces
- \( f(p_i) = 0 \) and \( \nabla f(p_i) = n_i \), \( \forall (p_i, n_i) \in D \)
- Is there a closed form solution?
- Function described in analytic form
- Function can be evaluated in finite time
- Is there a simple and elegant solution?

Oriented Convex Hull

\[ f_i(x) = n_i^t(x - p_i) \]

[Diagram: Supporting Linear Half Space]

\[ f_i(p_j) \leq 0 \]
Oriented Convex Hull

\[ f(x) = \max_i f_i(x) \]

Intersection of all Supporting Linear Half Spaces

Non-Convex Hull

\[ f_i(x) = n_i^t(x - p_i) - \frac{1}{2r_i} \|x - p_i\|^2 \]

\[ f(x) = \max_i f_i(x) \]

Max \( r_i \) so that \( f_i(p_j) \leq 0 \) for all \( j \)

Intersection of all Supporting Spherical Half Spaces
Properties of Non-Convex Hull Function

\[ f(x) = \max_i f_i(x) \]

\[ f_i(x) = n_i^t (x - p_i) - \frac{1}{2r_i} \| x - p_i \|^2 \]

Max \( r_i \) so that \( f_i(p_j) \leq 0 \) for all \( j \)

\[ f_i(p_i) = 0 \quad \nabla f_i(p_i) = n_i \]

\[ f_j(p_i) \leq 0 = f_i(p_i) \Rightarrow f(p_i) = f_i(p_i) = 0 \]

\[ p_i \in F_i = \{ x : f_i(x) > f_j(x) \forall j \neq i \} \text{ is open} \]

\[ \Rightarrow \nabla f(p_i) = \nabla f_i(p_i) = n_i \]

\[ \text{Voronoi and Apollonious Diagrams} \]
Geometry of the Support Functions

\[ f_i(x) = n_i^t (x - p_i) - \frac{1}{2r_i} \| x - p_i \|^2 \]

\[ q_i = p_i + r_i * n \]

\[ f_i(x) = \frac{1}{2r_i} \left( r_i^2 - \| x - q_i \|^2 \right) \]

NCH Surface Reconstruction

\[ f_i(x) = n_i^t (x - p_i) - \rho_i \| x - p_i \|^2 \]

\[ \rho_i = \frac{1}{2r_i} \quad 0 \leq \rho_i < \infty \]

\[ \rho_i = \min \left\{ \frac{n_i^t (p_j - p_i)}{\| p_j - p_i \|^2} : j \in J_i \right\} \]

\[ J_i = \left\{ j : n_i^t (p_j - p_i) > 0 \right\} \]

But \( \rho_i = 0 \) if \( J_i = \emptyset \)
procedure `estimateNCH()` {
    for i = 1 to i = N step 1 do {
        $\rho_i = 0$
        for j = 1 to j = N step 1 do {
            if $j = i$ continue
            $a = n_i^t(p_j - p_i)$
            $b = \|p_j - p_i\|^2$
            if ($a - \rho_i b > 0$) $\rho_i = a/b$
        }
    }
}

procedure `evaluateNCH(x)` {
    $f_x = -\infty$
    for i = 1 to i = N step 1 do {
        $a = n_i^t(x - p_i)$
        $b = \|x - p_i\|^2$
        $c = a - \rho_i b$
        if ($c > f_x$) $f_x = c$
    }
    return $f_x$
}
Another 2D Result

\[ C = \{ x : f(x) = 0 \} \]

Evaluate on pixel grid and compute isocurve

A 3D Example
Symmetric Non-Convex Hull

- If orientation of normal vectors is reversed, a different NCH Function results.
- Compute $f_i^+(x)$ from $\{(p_i, n_i) : i = 1, \ldots, N\}$
- Compute $f_i^-(x)$ from $\{(p_i, -n_i) : i = 1, \ldots, N\}$
- Define $f(x) = \frac{f_i^+(x) - f_i^-(x)}{2}$

NCH Surface Representation

- Set of oriented points with two additional scalar attributes

\[ \{(p_i, n_i, \rho_i^+, \rho_i^-) : i = 1, \ldots, N\} \]

- Can be saved as a PLY file
- Evaluate on tet-mesh vertices and compute piece-wise-linear isosurface
- Evaluate on dual vertices of octree and run Dual Marching Cubes
3D Results: evenly sampled low noise

3D Results: evenly sampled low noise
3D Results: unevenly sampled low noise

3D Results: unevenly sample and noise
Relation to the Medial Axis Transform

• The finite set of oriented points is replaced by the continuous boundary surface $S$ of a bounded solid object $O$, which is an open set in 3D
• The surface $S$ is smooth, with a continuous unit length normal field pointing towards the inside of $O$, and continuous curvatures.
• The Medial Axis Transform of $O$ is the family $\text{MAT}(O)$ of open balls contained in $O$ which are maximal with respect to inclusion.
• Inside, Outside, and Symmetric MAT

Medial Balls