DISCRETE SURFACE SIGNAL PROCESSING FOR FREE-FORM SURFACE DESIGN

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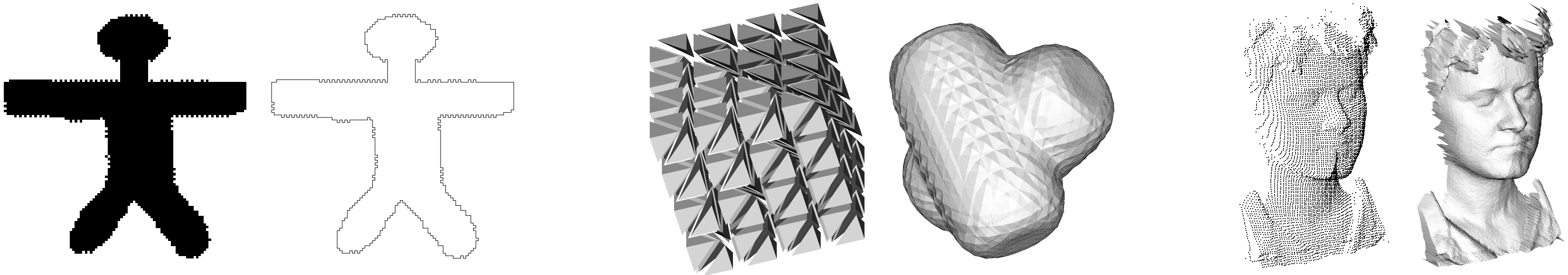
- CURVE AND SURFACE SMOOTHING WITHOUT SHRINKAGE ,
  by G. Taubin,
  Fifth International Conference on Computer Vision (ICCV'95).

- A SIGNAL PROCESSING APPROACH TO FAIR SURFACE DESIGN ,
  by G. Taubin,
  SIGGRAPH'95.

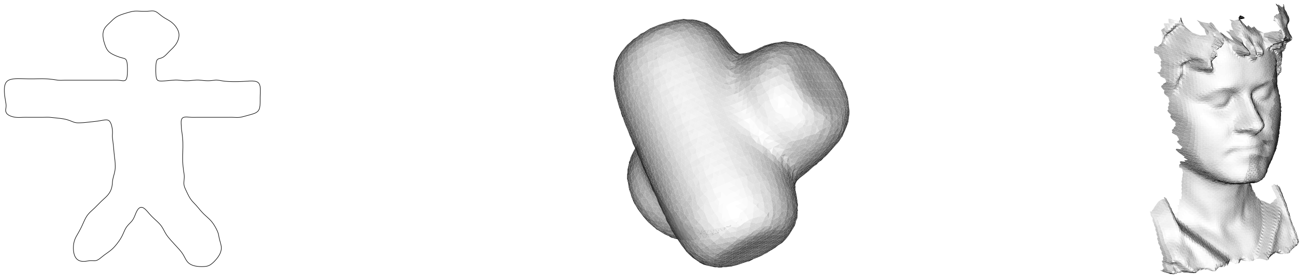
- FAST POLYHEDRAL SURFACE SMOOTHING ,
  by G. Taubin, T. Zhang (Stanford), and G. Golub (Stanford),
  (in preparation).

MOTIVATED BY THE PROBLEM OF SMOOTHING

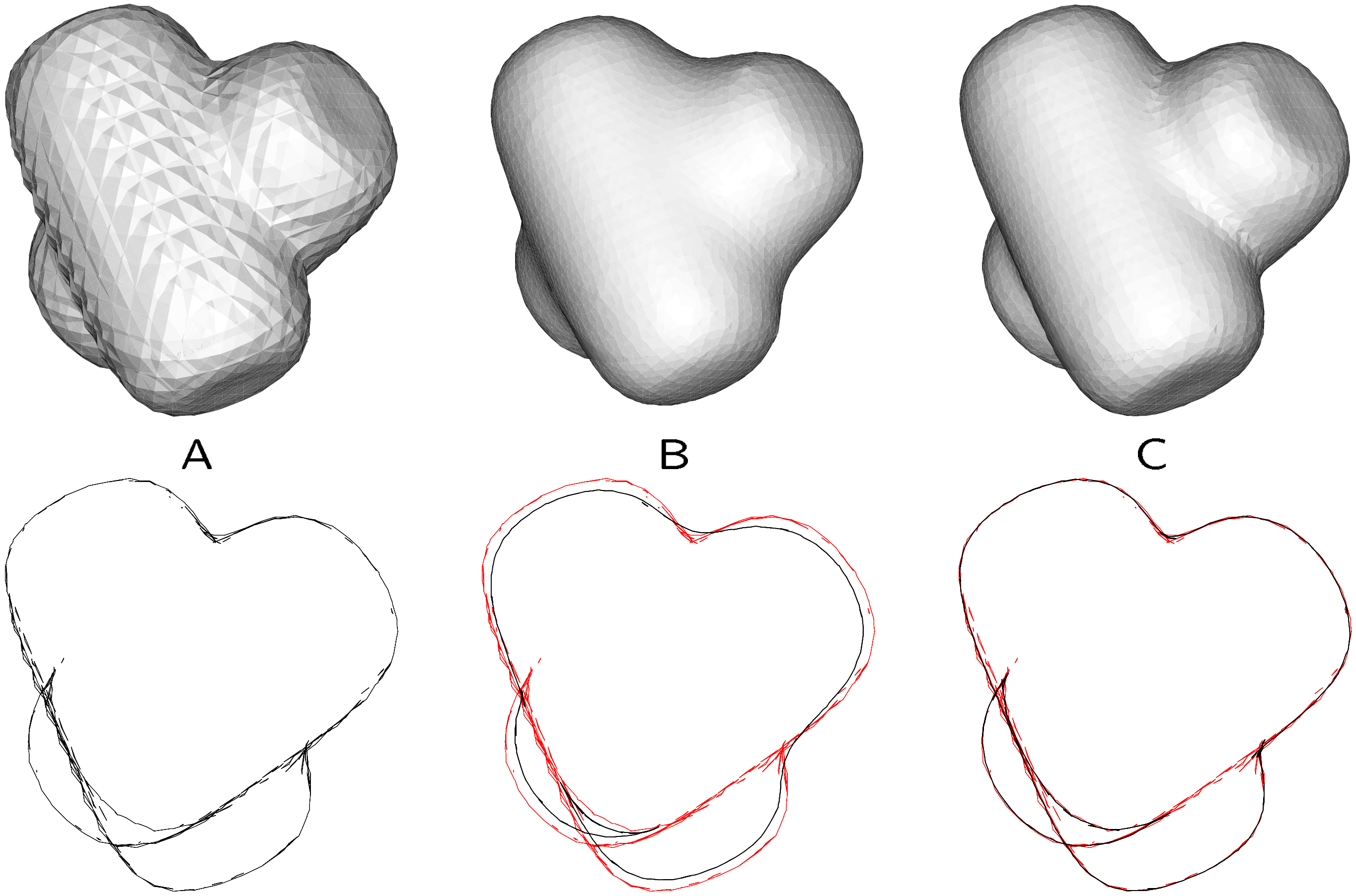
MANY ALGORITHMS (BOUNDARY FOLLOWING, ISO-SURFACES, ETC.), PRODUCE INACCURATE OR NOISY PIECE-WISE LINEAR APPROXIMATIONS OF CONTINUOUS CURVES AND SURFACES



HOW TO FORMULATE SURFACE SMOOTHING AS LOW-PASS FILTERING ?



EXAMPLE



A : An iso-surface appears faceted.

B : Gaussian smoothing.

C : Smoothing as low-pass filtering.

OVERVIEW

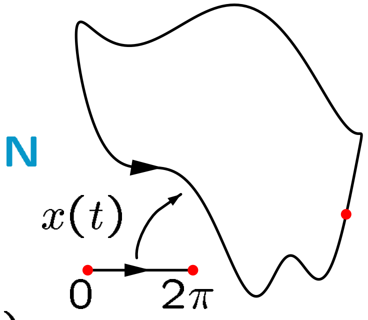
SURFACE SMOOTHING AS LOW-PASS FILTERING

- LOW-PASS FILTERING AS A LINEAR PROJECTION
- EXTENSION TO SIGNALS DEFINED ON SURFACES
- A SIMPLE SURFACE SIGNAL LOW-PASS FILTERING ALGORITHM
- FILTER DESIGN

APPLICATIONS TO FREE-FORM SURFACE DESIGN

- SUBDIVISION AND SMOOTHING
- INTERPOLATORY CONSTRAINTS
- SMOOTH INTERPOLATION

LOW-PASS FILTERING AS A LINEAR PROJECTION

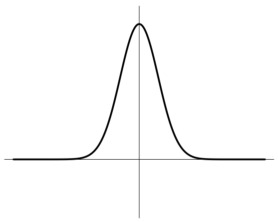


1) FOURIER DESCRIPTORS (COMPLEXITY $O(n\log(n))$)
COMPUTE FOURIER SERIES AND DISCARD TAIL

$$x(t) = \sum_{k=0}^{\infty} \xi_k\, u_k(t) \quad \mapsto \quad x'(t) = \sum_{k=0}^{k_{\text{SB}}} \xi_k\, u_k(t)$$

EXACT PROJECTION ONTO SUBSPACE OF LOW FREQUENCIES

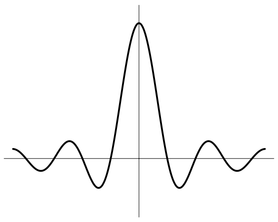
2) GAUSSIAN FILTERING (COMPLEXITY $O(n)$)
CONVOLVE WITH GAUSSIAN KERNEL



$$x(t) \quad \mapsto \quad x'(t) = \int g_\sigma(t-s)\, x(s)\, ds$$

NOT A LOW-PASS FILTER : PRODUCES SHRINKAGE

3) LOW-PASS FILTERING (COMPLEXITY $O(n)$)
CONVOLVE WITH LOW-PASS FILTER KERNEL



$$x(t) \quad \mapsto \quad x'(t) = \int k(t-s)\, x(s)\, ds$$

APPROX PROJECTION ONTO SUBSPACE OF LOW FREQUENCIES

FOURIER ANALYSIS AND THE LAPLACIAN

$x = (x_1, \ldots, x_n)^t$ DISCRETE TIME n-PERIODIC SIGNAL

1) THE DISCRETE LAPLACIAN OF x IS

$$\Delta x_i = \frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i)$$

2) GAUSSIAN SMOOTHING IS

$$x_i' = x_i + \lambda \, \Delta x_i \; .$$

OR IN MATRIX FORM

$$x' = (I - \lambda K)\, x \; ,$$

WHERE K IS THE CIRCULANT MATRIX

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix} \; .$$

3) LOW-PASS FILTERING IS

$$x' = f(K)\, x \; ,$$

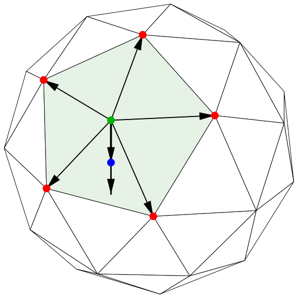
FOR SOME TRANSFER FUNCTION $f(k)$

EXTENSION TO SIGNALS DEFINED ON SURFACES

$x = (x_1, \ldots, x_n)^t$ FUNCTION DEFINED ON VERTICES OF SURFACE

1) REPLACE DISCRETE LAPLACIAN BY

$$\Delta x_i = \sum_{j \in i^\star} w_{ij} \, (x_j - x_i) \quad \text{WHERE} \quad w_{ij} > 0 \quad \sum_{j \in i^\star} w_{ij} = 1.$$



2) GAUSSIAN SMOOTHING IS STILL

$$x_i' = x_i + \lambda \, \Delta x_i \ ,$$

OR IN MATRIX FORM

$$x' = (I - \lambda K) \, x \ ,$$

WHERE $K = I - W$ IS NO LONGER A CIRCULANT MATRIX

3) LOW-PASS FILTERING IS STILL

$$x' = f(K) \, x \ ,$$

FOR SOME TRANSFER FUNCTION $f(k)$

WEIGHT MATRIX

$$W = (\,w_{ij}\,)$$

SYMMETRIC NEIGHBORHOOD STRUCTURE $\Rightarrow$ W IS NORMAL $\Rightarrow$

W HAS REAL EIGENVALUES

$\sum_{j \in i^\star} w_{ij} = 1 \Rightarrow W$ IS STOCHASTIC $\Rightarrow$

EIGENVALUES OF W IN $\{z : |z| \leq 1\}$

EIGENVALUES AND RIGHT EIGENVECTORS OF $K = I - W$

$0 \leq k_1 \leq \cdots \leq k_{n_V} \leq 2$ ← NATURAL FREQUENCIES

$u_1, \ldots, u_{n_V}$ ← NATURAL VIBRATION MODES

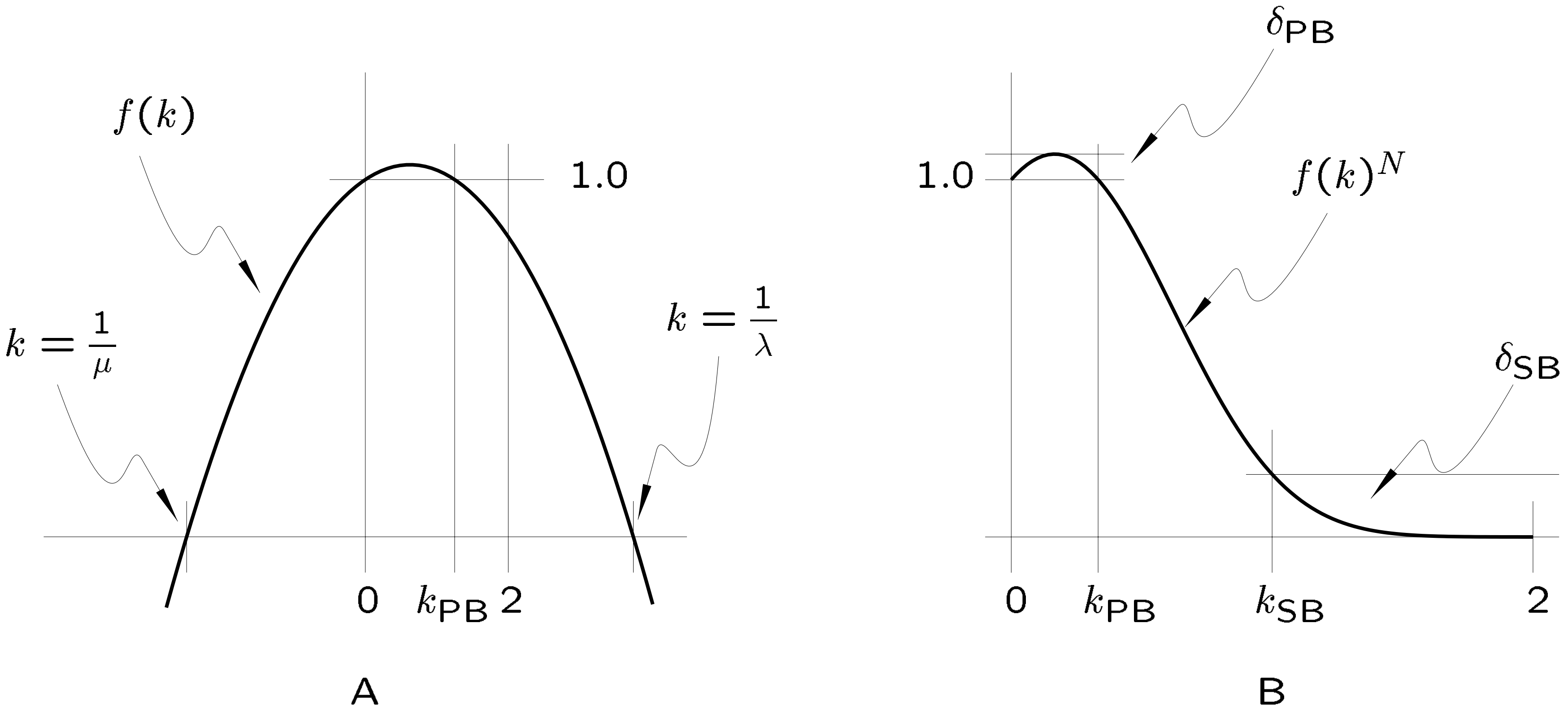
IF $f(k)$ POLYNOMIAL TRANFER FUNCTION $\Rightarrow f(K)u_i = f(k_i)u_i \Rightarrow$

$$x' = f(K)x = \sum_{i=1}^{n} f(k_i)\,\xi_i\,u_i$$

FOR GAUSSIAN SMOOTHING $f(k) = (1 - \lambda k)$ WITH $0 < \lambda < 1/2$
THIS IS NOT A LOW-PASS FILTER

FOR NEW ALGORITHM $f(k) = (1 - \mu k)(1 - \lambda k)$ WITH $0 < \lambda < -\mu$
THIS IS A LOW-PASS FILTER

NEW SMOOTHING ALGORITHM IS A LOW-PASS FILTER



A : Graph of the polynomial $f(k) = (1 - \lambda\, k)(1 - \mu\, k)$.

B : Graph of the transfer function $f(k)^N$.

$$0 < \lambda < -\mu \quad \Rightarrow \quad k_{\text{PB}} = \frac{1}{\lambda} + \frac{1}{\mu} > 0$$

FILTER DESIGN (1)

(WITH GENE GOLUB AND TONG ZHANG, STANFORD)

- WE LOOK FOR OPTIMAL POLYNOMIAL APPROXIMATION OF

$$f_{\mathsf{LP}}(k) = \begin{cases} 1 & \text{if} \quad 0 \leq k < k_{\mathsf{PB}} \\ 0 & \text{if} \quad k_{\mathsf{PB}} \leq k < \&2 \end{cases} ,$$

- CHANGE VARIABLES $k = 2(1 - \cos(\theta))$ AND EXPAND

$$h_{\mathsf{LP}}(\theta) = h_0 + 2\sum_{n=0}^{\infty} h_n \cos(n\theta) = (\theta_{\mathsf{PB}}/\pi) + \sum_{n=0}^{\infty} (2\sin(n\,\theta_{\mathsf{PB}})/n\,\pi)\cos(n\theta) .$$

- USE CHEBYSHEV POLYNOMIALS $\cos(n\,\theta) = T_n(\cos(\theta))$

$$T_n(w) = \begin{cases} 1 & n = 0 \\ w & n = 1 \\ 2\,w\,T_{n-1}(w) - T_{n-2}(w) & n > 1 \end{cases}$$

TO GET APPROXIMATION IN ORIGINAL VARIABLE

$$f_N(k) = (\theta_{\mathsf{PB}}/\pi)\,T_0(1 - k/2) + \sum_{n=1}^{N} (2\sin(n\,\theta_{\mathsf{PB}})/n\,\pi)\,T_n(1 - k/2) .$$

FILTER DESIGN (2)

- TO ATTENUATE GIBBS PHENOMENON USE WEIGHTS

$$f_N(k) = w_0\,(\theta_{\mathrm{PB}}/\pi)\,T_0(1-k/2) + \sum_{n=1}^{N} w_n\,(2\sin(n\,\theta_{\mathrm{PB}})/n\,\pi)\,T_n(1-k/2)\ ,$$

- RECTANGULAR WINDOW

$$w_n = 1\ .$$

- HANNING WINDOW

$$w_n = 0.5 + 0.5\cos(n*\pi/(N+1))\ .$$

- HAMMING WINDOW

$$w_n = 0.54 + 0.46\cos(n*\pi/(N+1))\ .$$

- BLACKMAN WINDOW WINDOW

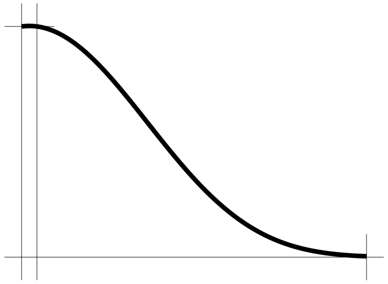
$$w_n = 0.42 + 0.5\cos(n\pi/(N+1)) + 0.08\cos(2n\pi/(N+1))\ .$$

- OTHER FIR DIGITAL FILTER DESIGN TECHNIQUES: EQUIRIPPLE FILTERS, MAXIMALLY FLAT FILTERS, ETC.

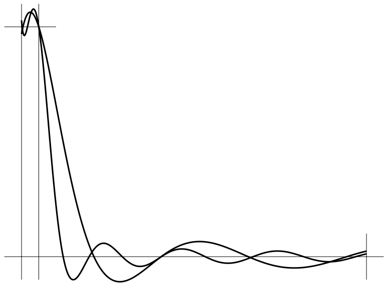
IF TOO NARROW BAND-PASS REGION

$$f_N(k) = w_0 \frac{\theta_{\mathrm{PB}}+\sigma}{\pi} T_0(1-k/2) + \sum_{n=1}^{N} w_n \frac{2\sin(n(\theta_{\mathrm{PB}}+\sigma))}{n\,\pi} T_n(1-k/2)\,,$$

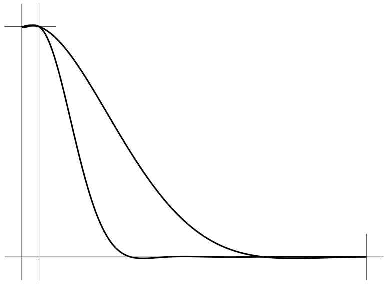
$((1-\lambda k)(1-\mu k))^N$
$N = 10$



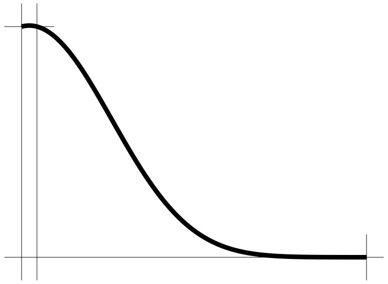
RECTANGULAR
$N = 10, 20$



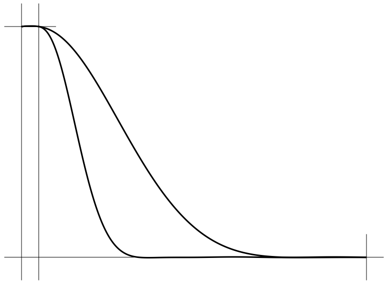
HANNING
$N = 10, 20$



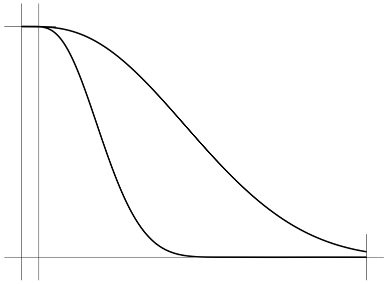
$((1-\lambda k)(1-\mu k))^N$
$N = 20$



HAMMING
$N = 10, 20$



BLACKMAN
$N = 10, 20$



PARTIAL SUMMARY

A NEW METHOD FOR SMOOTHING PIECE-WISE LINEAR CURVES AND SURFACES

- APPLIES TO PIECE-WISE LINEAR SHAPES OF ANY DIMENSION AND TOPOLOGY
- ITS COMPUTATIONAL COMPLEXITY IS LINEAR BOTH IN TIME AND IN SPACE
- PRODUCES LOW-PASS FILTER EFFECT AS A FUNCTION OF CURVATURE
- DOES NOT PRODUCE SHRINKAGE
- IT IS VERY SIMPLE TO IMPLEMENT

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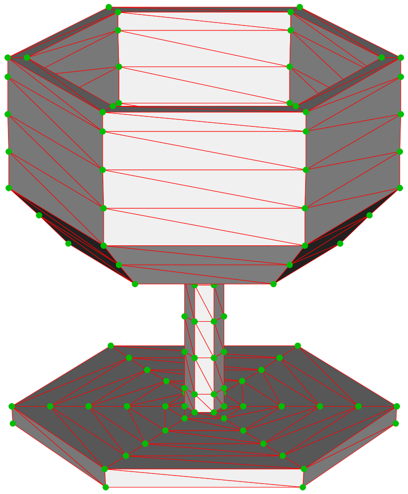
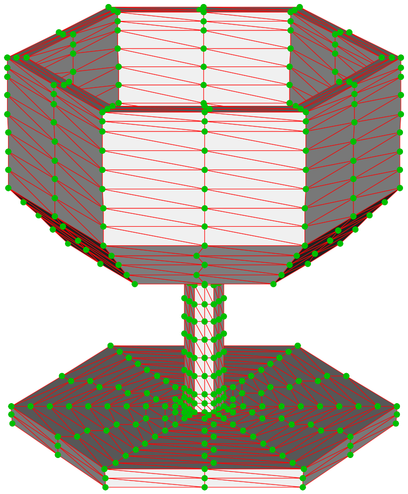
FREE-FORM SURFACE DESIGN: SUBDIVISION + SMOOTHING (1)

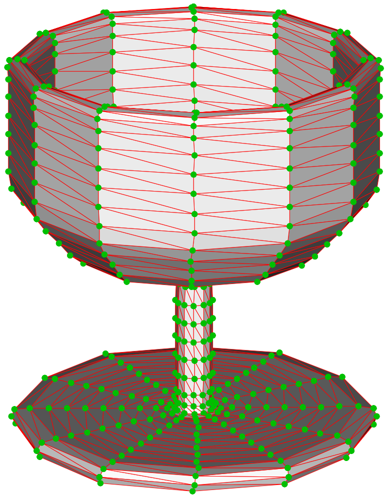
APPLY SUBDIVISION AND SMOOTHING STEPS



SKELETON (S_0)



SUBDIVIDED

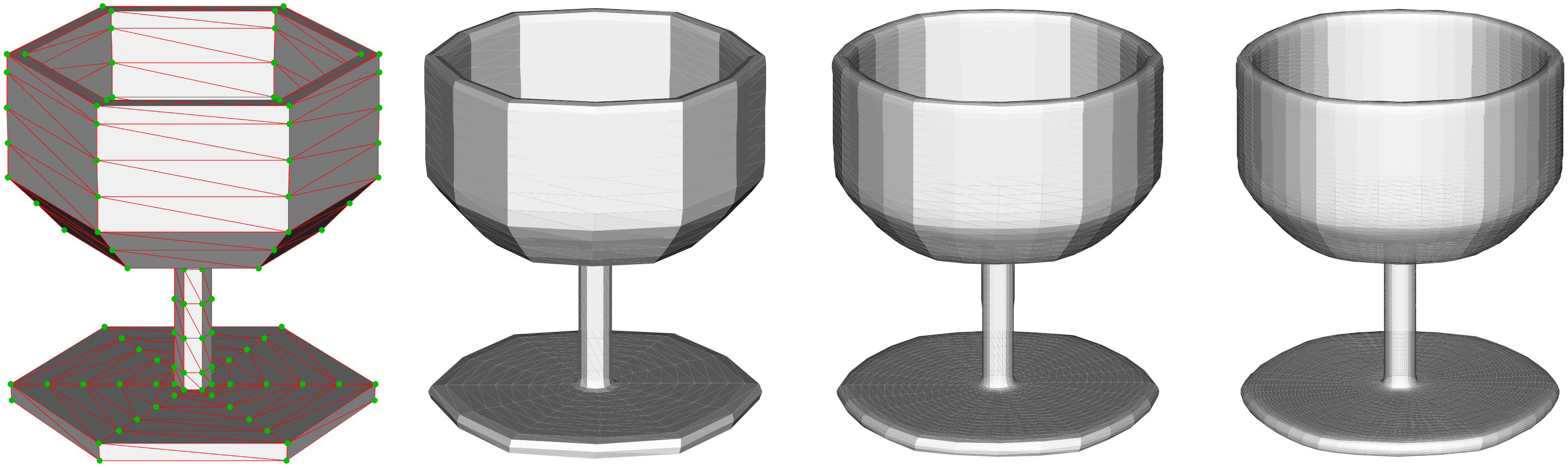


SMOOTHED (S_1)

USUALLY ONLY ONE STEP OF GAUSSIAN SMOOTHING WITH $\lambda = 0.5$

FREE-FORM SURFACE DESIGN: SUBDIVISION + SMOOTHING (2)

ONE STEP OF GAUSSIAN SMOOTHING WITH $\lambda = 0.5$



NOT ENOUGH SMOOTHING :
HEXAGONAL SYMMETRY OF SKELETON REMAINS

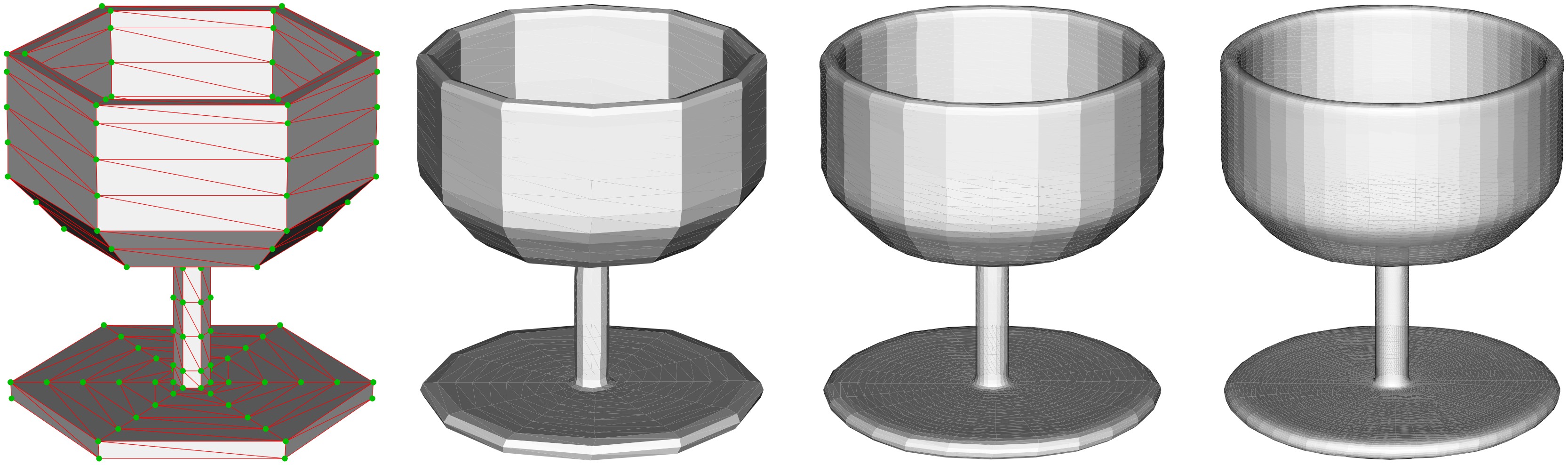
FREE-FORM SURFACE DESIGN: SUBDIVISION + SMOOTHING (3)

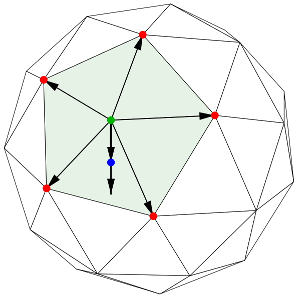
10 STEPS OF GAUSSIAN SMOOTHING WITH $\lambda = 0.5$

20 STEPS OF GAUSSIAN SMOOTHING WITH $\lambda = 0.5$

FREE-FORM SURFACE DESIGN: SUBDIVISION + SMOOTHING (4)

20 STEPS OF NEW ALGORITHM WITH $\lambda = 0.33$ $\mu = 0.34$





INTERPOLATORY CONSTRAINTS (1)

$x = (x_1, \dots, x_n)^t$ FUNCTION DEFINED ON VERTICES OF SURFACE

USE NON-SYMMETRIC NEIGHBORHOODS

1) TO FIX A VERTEX MAKE ITS NEIGHBORHOOD EMPTY

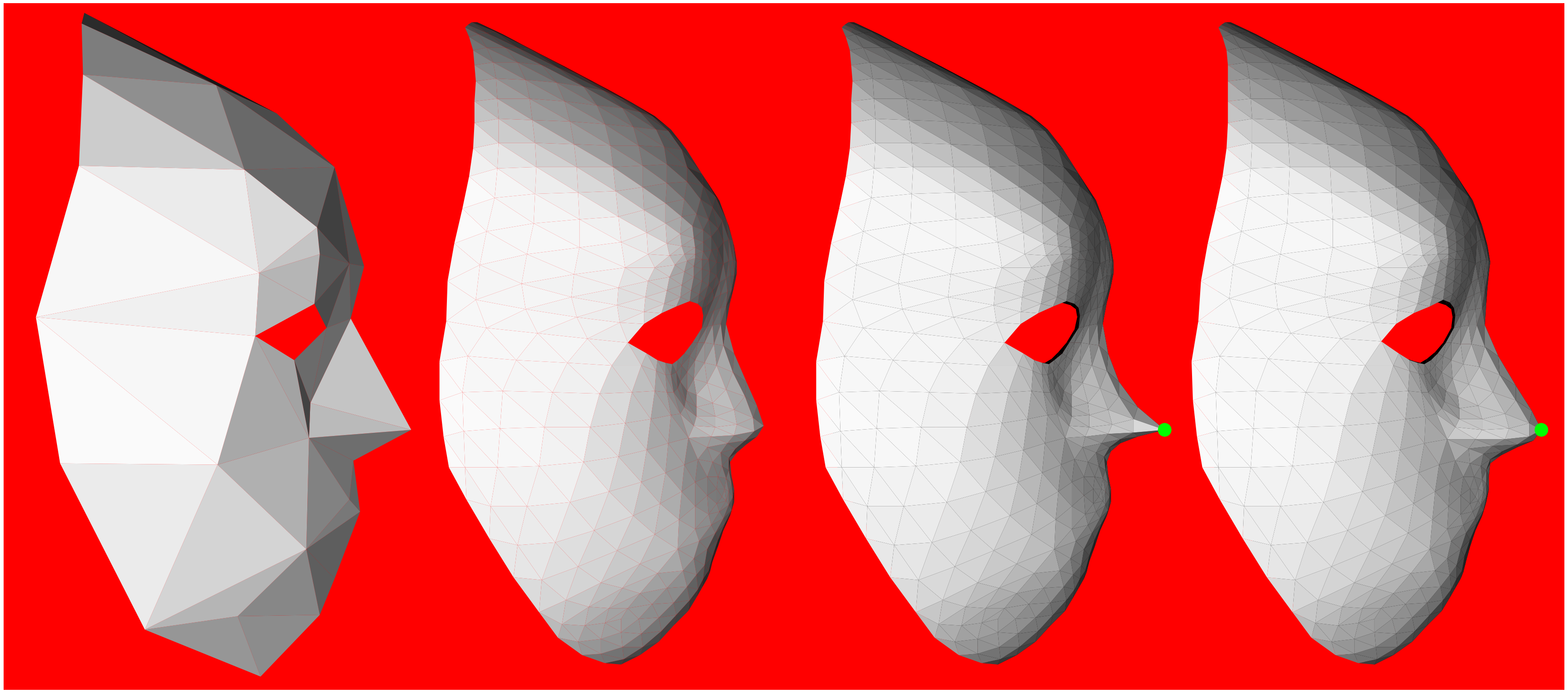
SMOOTHNESS IS LOST AT THE VERTEX

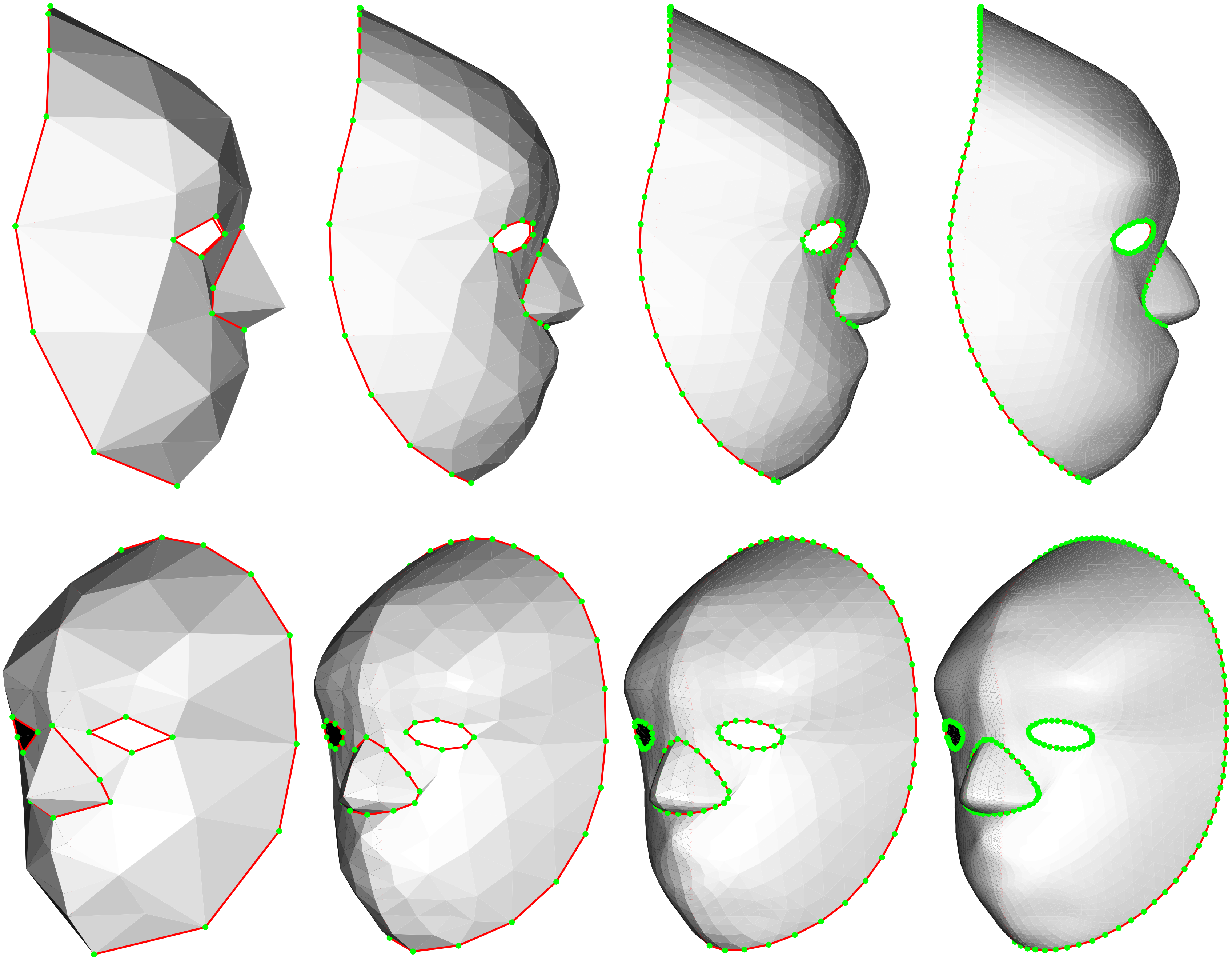
2) TO SMOOTH A SURFACE WITH BOUNDARY DO NOT MAKE INTERNAL VERTICES NEIGHBORS OF BOUNDARY VERTICES

CAN BE USED TO DESIGN SURFACES WITH INTERNAL RIDGE CURVES

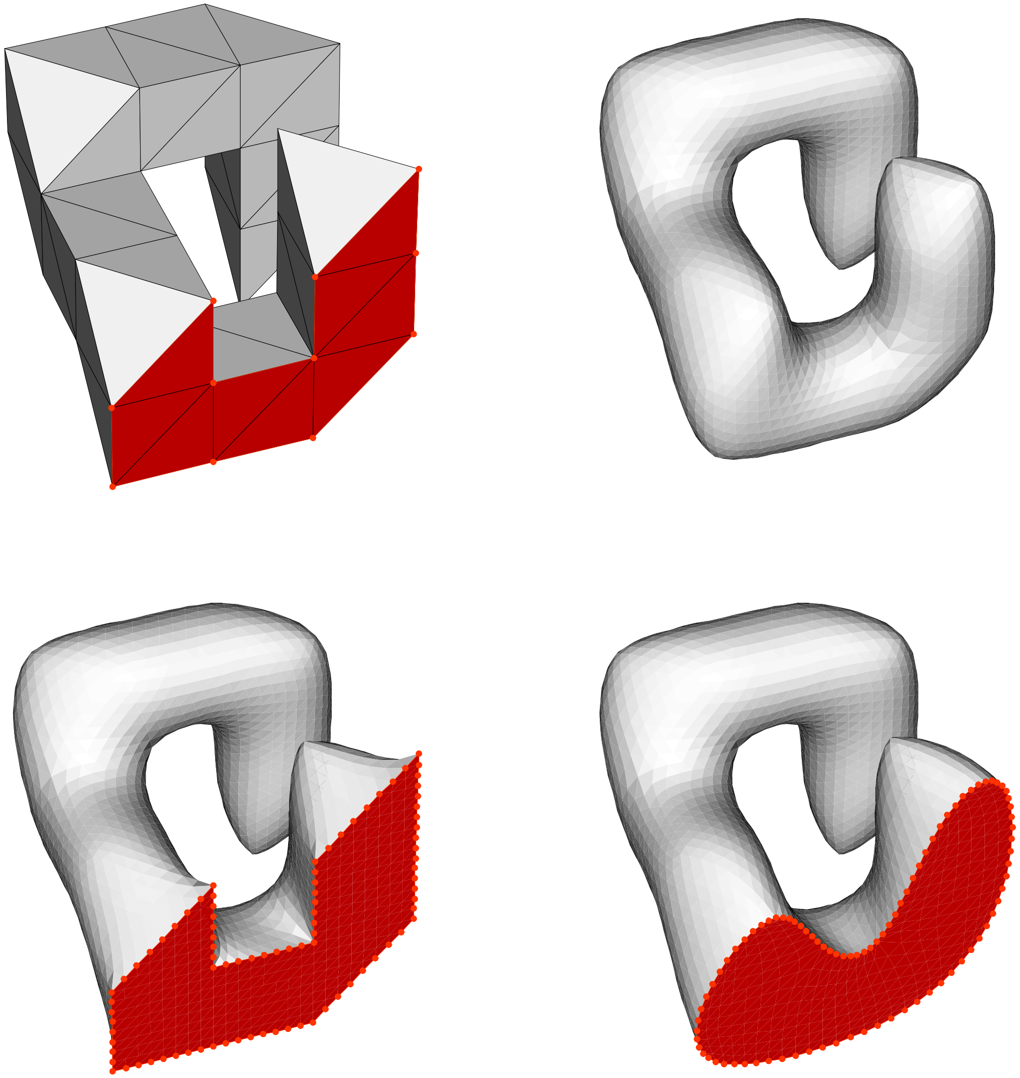
3) HIERARCHICAL NEIGHBORHOODS:
ASSIGN A NUMERIC LABEL l_i TO EACH VERTEX v_i AND
IF $l_j > l_i$ DO NOT CONSIDER v_j A NEIGHBOR OF v_i

INTERPOLATORY CONSTRAINTS (2)

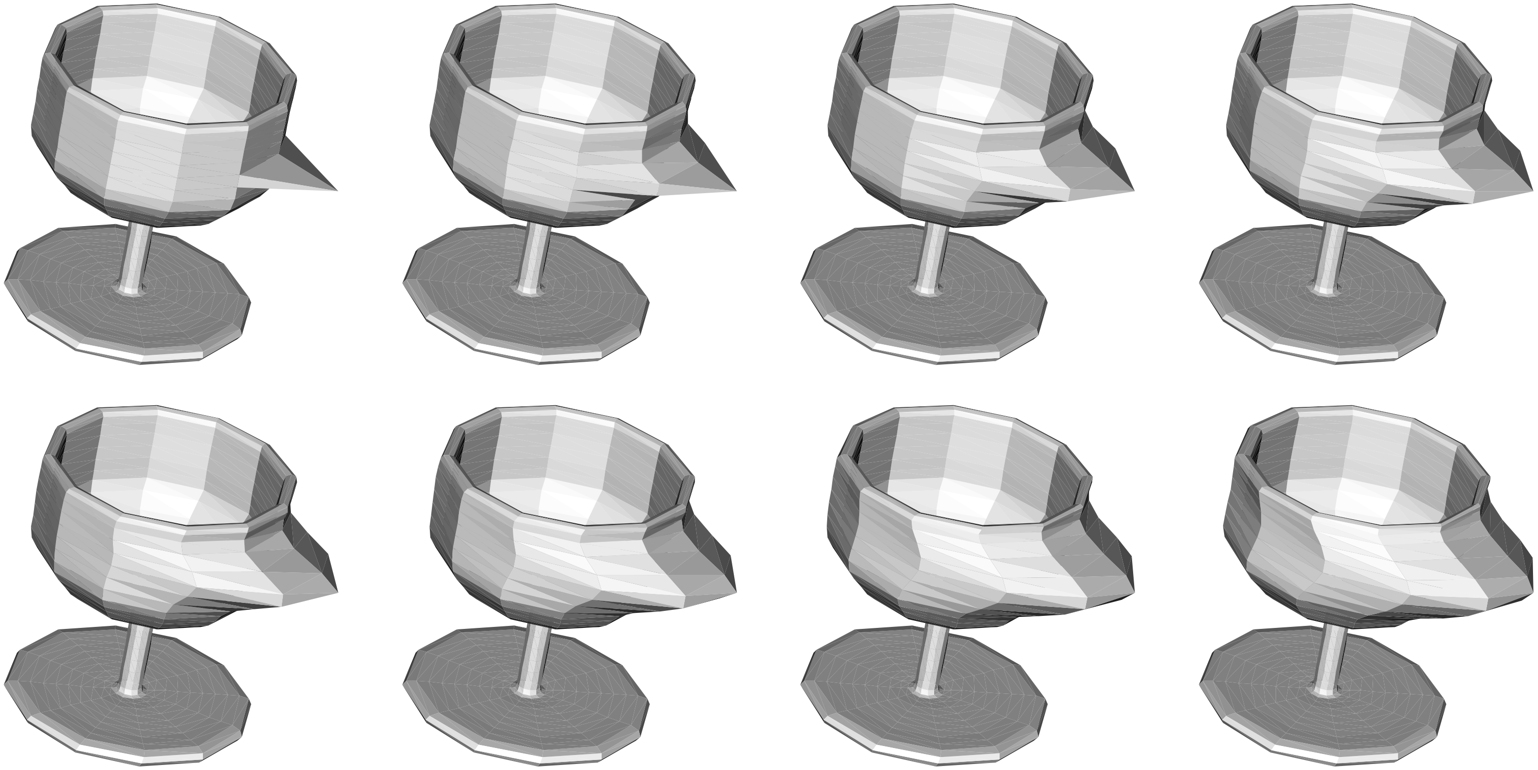




HIERARCHICAL CONSTRAINTS



SMOOTH INTERPOLATION (1)



SMOOTH INTERPOLATION (2)

ONE CONSTRAINT $(x_C^N)_1 = x_1$;

- WRITE DESIRED CONSTRAINED SMOOTH SIGNAL x_C^N AS SUM OF UNCONSTRAINED SMOOTH SIGNAL $x^N = F\,x$ $(F = f(K)^N)$ PLUS SMOOTH DEFORMATION d_1

$$x_C^N = x^N + (x_1 - x_1^N)\, d_1 \ .$$

  DEFORMATION d_1 IS ITSELF ANOTHER DISCRETE SURFACE SIGNAL, AND THE CONSTRAINT $(x_C^N)_1 = x_1$ IS SATISFIED IF $(d_1)_1 = 1$.

- DEFORMATION d_1 IS CONSTRUCTED BY APPLYING SMOOTHING ALGORITHM TO δ_1

$$(\delta_i)_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \ ,$$

  AND THEN RESCALING THE RESULT TO MAKE IT SATISFY THE CONSTRAINT

$$d_1 = F_{n1} F_{11}^{-1} \ .$$

F_{rs} DENOTES THE SUB-MATRIX OF $F = f(K)^N$ DETERMINED BY THE FIRST r ROWS AND THE FIRST s COLUMNS.

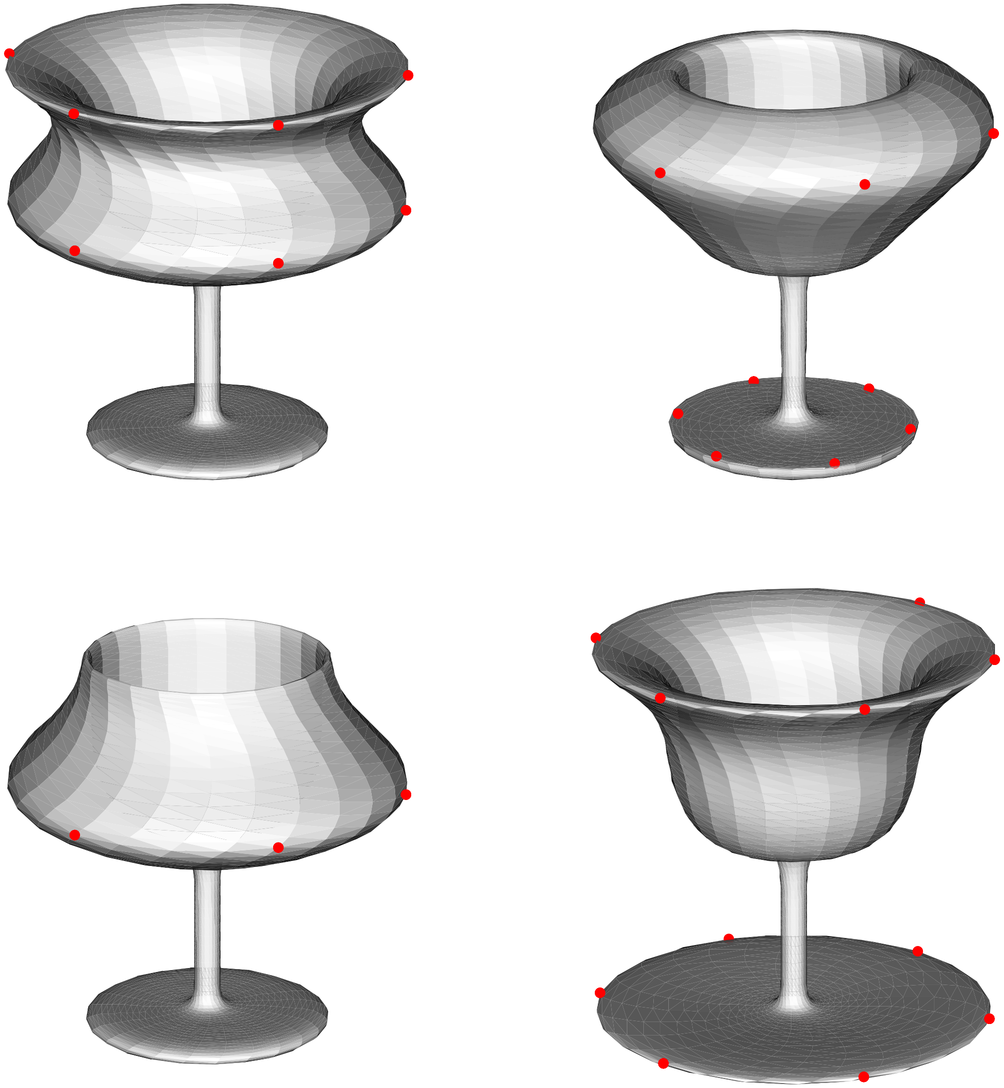
SMOOTH INTERPOLATION (2)

SEVERAL CONSTRAINTS $(x_C^N)_1 = x_1, \ldots, (x_C^N)_m = x_m$

$$x_C^N = x^N + F_{nm}\, F_{mm}^{-1} \begin{pmatrix} x_1 - x_1^N \\ \vdots \\ x_m - x_m^N \end{pmatrix} .$$

F_{rs} DENOTES THE SUB-MATRIX OF $F = f(K)^N$ DETERMINED BY THE FIRST r ROWS AND THE FIRST s COLUMNS.

SUBDIVISION + SMOOTHING + SMOOTH INTERPOLATION



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- IT IS VERY SIMPLE TO IMPLEMENT
- SIMPLE MODIFICATIONS MAKE IT SATISFY DIFFERENT TYPES OF CONSTRAINTS