AM 255: Problem Set 1

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Problem 1

Let f be a real function with the Fourier series

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}.$$

Prove that

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N}^{N} \hat{f}(\omega) e^{i\omega x}$$

is real for all N.

First, we note that S_N is real for all N if and only if $S_N = \overline{S}_N$ (i.e., S_N is equal to its complex conjugate $\overline{S}_N$).

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N}^{N} \hat{f}(\omega) e^{i\omega x} \tag{1}$$

$$\overline{S}_N = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N}^{N} \overline{\hat{f}(\omega)} e^{-i\omega x} = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-N}^{N} \overline{\hat{f}(-\omega)} e^{i\omega x} \tag{2}$$

Note that, on the right-hand side (RHS) of Equation 2, we have substituted $\omega \to -\omega$ (since the summation can be taken in any order).

Equating Equations 1 and 2, we find that $\hat{f}(\omega)$ must be *conjugate symmetric* in order for S_N to be a real-valued function for all N (i.e., $\hat{f}(\omega) = \overline{\hat{f}(-\omega)}$). Recall from Equation 1.1.2 in [1], we have

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-i\omega x} f(x) dx = \frac{1}{\sqrt{2\pi}} \, (e^{i\omega x}, f(x)), \tag{3}$$

where on the RHS we have used the definition of the L_2 scalar product norm given by

$$(f, g) = \int_0^{2\pi} \overline{f} g \; dx.$$

Given these definitions, we have

$$\begin{aligned} \hat{f}(-\omega) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{i\omega x} f(x) dx = \frac{1}{\sqrt{2\pi}} \, (e^{-i\omega x}, f(x)) \\ \Rightarrow \overline{\hat{f}(-\omega)} &= \frac{1}{\sqrt{2\pi}} \overline{\int_0^{2\pi} e^{i\omega x} f(x) dx} = \frac{1}{\sqrt{2\pi}} \, \overline{(e^{-i\omega x}, f(x))} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-i\omega x} \overline{f(x)} dx = \frac{1}{\sqrt{2\pi}} \, (e^{i\omega x}, \overline{f(x)}). \end{aligned} \tag{4}$$

Since f is a real function we must have $f(x) = \overline{f(x)}$. Equating Equations 3 and 4, we prove the desired result: if f is a real function, then its Fourier coefficients $\hat{f}(\omega)$ are conjugate symmetric such that $\hat{f}(\omega) = \overline{\hat{f}(-\omega)}$ and, as a result, $S_N = \overline{S}_N$ and S_N must be real for all N.

$$\boxed{\hat{f}(\omega) = \overline{\hat{f}(-\omega)} \;\Rightarrow\; S_N = \overline{S}_N, \;\; \therefore S_N \text{ is real for all } N} \tag{5}$$

(QED)

Problem 2

Derive estimates for

$$\left|\left(D - \frac{\partial^3}{\partial x^3}\right) e^{i\omega x}\right|$$

where $D = D_+^3,\ D_-D_+^2,\ D_-^2D_+,\ D_-^3,\ D_0D_+D_-$.

In this problem we will investigate the approximation of the third partial derivative $\partial^3/\partial x^3$ using several difference operators. To begin our analysis, we evaluate the result analytically.

$$\frac{\partial^3}{\partial x^3} e^{i\omega x} = -i\omega^3 e^{i\omega x} \tag{6}$$

The following Taylor series expansions will also be required in the subsequent analysis.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \mathcal{O}(x^2)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \mathcal{O}(x^5)$$

Finally, we repeat for completeness the forward, backward, and central difference operators in terms of the shift operator E.

$$D_+ = (E - E^0)/h,\;\; D_- = (E^0 - E^{-1})/h, \;\text{and}\; D_0 = (E - E^{-1})/2h$$

Part (a): $D = D_+^3$

Substituting for the forward difference operator we find

$$\begin{aligned} D_+^3 e^{i\omega x} &= D_+^2 D_+ e^{i\omega x} = h^{-1} D_+^2 (E - E^0) e^{i\omega x} = h^{-1}(e^{i\omega h} - 1) D_+^2 e^{i\omega x} \\ &= h^{-2}(e^{i\omega h} - 1)^2 D_+ e^{i\omega x} = h^{-3}(e^{i\omega h} - 1)^3 e^{i\omega x}. \end{aligned}$$

At this point we can substitute the Taylor series expansion for $e^{i\omega h}$ such that

$$D_+^3 e^{i\omega x} = h^{-3}(i\omega h + \mathcal{O}(\omega^2 h^2))^3 e^{i\omega x} = (-i\omega^3 + \mathcal{O}(\omega^4 h)) e^{i\omega x}.$$

In conclusion, we find that D_+^3 is a first-order accurate approximation of $\partial^3/\partial x^3$ since the error is proportional to h.

$$\boxed{\left|\left(D_+^3 - \frac{\partial^3}{\partial x^3}\right) e^{i\omega x}\right| = \mathcal{O}(\omega^4 h)} \tag{7}$$

Part (b): $D = D_- D_+^2$

Substituting for the forward and backward difference operators we find

$$\begin{aligned} D_- D_+^2 e^{i\omega x} &= D_- D_+ D_+ e^{i\omega x} = h^{-1} D_- D_+ (E - E^0) e^{i\omega x} = h^{-1}(e^{i\omega h} - 1) D_- D_+ e^{i\omega x} \\ &= h^{-2}(e^{i\omega h} - 1)^2 D_- e^{i\omega x} = h^{-3}(e^{i\omega h} - 1)^2 (1 - e^{-i\omega h}) e^{i\omega x}. \end{aligned}$$

At this point we can substitute the Taylor series expansions for $e^{\pm i\omega h}$ such that

$$D_- D_+^2 e^{i\omega x} = h^{-3}(i\omega h + \mathcal{O}(\omega^2 h^2))^2 (i\omega h + \mathcal{O}(\omega^2 h^2)) e^{i\omega x} = (-i\omega^3 + \mathcal{O}(\omega^4 h)) e^{i\omega x}.$$

In conclusion, we find that $D_- D_+^2$ is a first-order accurate approximation of $\partial^3/\partial x^3$ since the error is proportional to h.

$$\boxed{\left| \left(D_- D_+^2 - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| = \mathcal{O}(\omega^4 h)} \tag{8}$$

Part (c): $D = D_-^2 D_+$

Substituting for the forward and backward difference operators we find

$$\begin{aligned} D_-^2 D_+ e^{i\omega x} &= h^{-1} D_-^2 (E - E^0) e^{i\omega x} = h^{-1}(e^{i\omega h} - 1) D_- D_- e^{i\omega x} \\ &= h^{-2}(e^{i\omega h} - 1)(1 - e^{-i\omega h}) D_- e^{i\omega x} = h^{-3}(e^{i\omega h} - 1)(1 - e^{-i\omega h})^2 e^{i\omega x}. \end{aligned}$$

At this point we can substitute the Taylor series expansions for $e^{\pm i\omega h}$ such that

$$D_-^2 D_+ e^{i\omega x} = h^{-3}(i\omega h + \mathcal{O}(\omega^2 h^2))(i\omega h + \mathcal{O}(\omega^2 h^2))^2 e^{i\omega x} = (-i\omega^3 + \mathcal{O}(\omega^4 h)) e^{i\omega x}.$$

In conclusion, we find that $D_-^2 D_+$ is a first-order accurate approximation of $\partial^3/\partial x^3$ since the error is proportional to h.

$$\boxed{\left| \left(D_-^2 D_+ - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| = \mathcal{O}(\omega^4 h)} \tag{9}$$

Part (d): $D = D_-^3$

Substituting for the backward difference operator we find

$$\begin{aligned} D_-^3 e^{i\omega x} &= h^{-1} D_-^2 (E^0 - E^{-1}) e^{i\omega x} = h^{-1}(1 - e^{-i\omega h}) D_- D_- e^{i\omega x} \\ &= h^{-2}(1 - e^{-i\omega h})^2 D_- e^{i\omega x} = h^{-3}(1 - e^{-i\omega h})^3 e^{i\omega x}. \end{aligned}$$

At this point we can substitute the Taylor series expansions for $e^{-i\omega h}$ such that

$$D_-^3 e^{i\omega x} = h^{-3}(i\omega h + \mathcal{O}(\omega^2 h^2))^3 e^{i\omega x} = (-i\omega^3 + \mathcal{O}(\omega^4 h)) e^{i\omega x}.$$

In conclusion, we find that D_-^3 is a first-order accurate approximation of $\partial^3/\partial x^3$ since the error is proportional to h.

$$\boxed{\left| \left(D_-^3 - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| = \mathcal{O}(\omega^4 h)} \tag{10}$$

Part (e): $D = D_0 D_+ D_-$

Substituting for the forward, backward, and central difference operators we find

$$\begin{aligned}
D_0 D_+ D_- e^{i\omega x} &= h^{-1} D_0 D_+ (E^0 - E^{-1}) e^{i\omega x} = h^{-1}(1 - e^{-i\omega h}) D_0 D_+ e^{i\omega x} \\
&= h^{-2}(1 - e^{-i\omega h})(e^{i\omega h} - 1) D_0 e^{i\omega x} \\
&= i h^{-3}(1 - e^{-i\omega h})(e^{i\omega h} - 1) \left(\frac{e^{i\omega h} - e^{-i\omega h}}{2i} \right) e^{i\omega x} \\
&= i h^{-3}(1 - e^{-i\omega h})(e^{i\omega h} - 1) \sin(\omega h) e^{i\omega x}.
\end{aligned}$$

At this point we can substitute the Taylor series expansions for $e^{\pm i\omega h}$ and $\sin(\omega h)$ such that

$$\begin{aligned}
D_0 D_+ D_- e^{i\omega x} &= i h^{-3}(i\omega h + \mathcal{O}(\omega^2 h^2))(i\omega h + \mathcal{O}(\omega^2 h^2))(\omega h + \mathcal{O}(\omega^3 h^3)) e^{i\omega x} \\
&= (-i\omega^3 + \mathcal{O}(\omega^5 h^2)) e^{i\omega x}.
\end{aligned}$$

In conclusion, we find that $D_0 D_+ D_-$ is a second-order accurate approximation of $\partial^3/\partial x^3$ since the error is proportional to h^2.

$$\boxed{\left| \left(D_0 D_+ D_- - \frac{\partial^3}{\partial x^3} \right) e^{i\omega x} \right| = \mathcal{O}(\omega^5 h^2)} \tag{11}$$

Problem 3

Compute $\|D_+ D_-\|_h$.

We will begin by deriving an upper bound on the discrete norm for periodic gridfunctions. Given two operators P and Q, Equation 1.2.12 in [1] gives

$$\|PQ\|_h \leq \|P\|_h \|Q\|_h.$$

Also from page 22 in [1], we have

$$\|D_+\|_h \leq 2/h \text{ and } \|D_-\|_h \leq 2/h.$$

Substituting, we find the following upper bound for the discrete operator norm of $D_+ D_-$.

$$\|D_+ D_-\|_h \leq \|D_+\|_h \|D_-\|_h \leq 4/h^2 \tag{12}$$

We can prove that the inequality in Equation 12 can be replaced with an equality by selecting a specific periodic gridfunction u. Let us begin by substituting for the forward and backward difference operators in terms of the shift operator E.

$$D_+ = (E - E^0)/h \text{ and } D_- = (E^0 - E^{-1})/h$$

Substituting for D_+ and D_- we find the following result.

$$(D_+ D_- u)_j = h^{-2}((E - 2E^0 + E^{-1})u)_j = h^{-2}(u_{j+1} - 2u_j + u_{j-1})$$

The definition of the discrete scalar product and norm for periodic gridfunctions are given as follows.

$$(u,v)_h = \sum_{j=0}^{N} \overline{u}_j v_j h \text{ and } \|u\|_h^2 = (u,u)_h$$

As a result, we have

$$\begin{aligned}
\|D_+D_-u\|_h^2 &= \sum_{j=0}^{N} \left(\frac{\overline{u}_{j+1} - 2\overline{u}_j + \overline{u}_{j-1}}{h^2}\right)\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right) h \\
&= \frac{1}{h^3}\sum_{j=0}^{N} (\overline{u}_{j+1} - 2\overline{u}_j + \overline{u}_{j-1})(u_{j+1} - 2u_j + u_{j-1}).
\end{aligned}$$

Motivated by the example on page 23 of [1], we define $u_j = (-1)^j$ such that

$$\|u\|_h^2 = (N+1)h \tag{13}$$

$$\begin{aligned}
\|D_+D_-u\|_h^2 &= \frac{1}{h^3}\sum_{j=0}^{N} ((-1)^{j+1} - 2(-1)^j + (-1)^{j-1})^2 \\
&= \frac{1}{h^3}\sum_{j=0}^{N} 16 = \frac{16(N+1)}{h^3} = \frac{16}{h^4}\ \|u\|_h^2
\end{aligned}$$

$$\Rightarrow \|D_+D_-u\|_h = \frac{4}{h^2}\ \|u\|_h. \tag{14}$$

Recall that the discrete norm of an operator Q is given by

$$\|Q\|_h = \sup_{u \neq 0} \|Qu\|_h / \|u\|_h.$$

Combining Equations 13 and 14 we find

$$\|D_+D_-u\|_h / \|u\|_h = 4/h^2, \text{ for } u_j = (-1)^j. \tag{15}$$

In conclusion, we have found a "witness" gridfunction $u_j = (-1)^j$ which achieves the upper bound in Equation 12. By the definition of the discrete operator norm, we conclude

$$\boxed{\|D_+D_-\|_h = 4/h^2} \tag{16}$$

References

[1] Bertil Gustafsson, Heinz-Otto Kreiss, and Joseph Oliger. *Time Dependent Problems and Difference Methods.* John Wiley & Sons, 1995.