AM 255: Problem Set 3

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Problem 1

Prove that the θ *scheme*

$$(I - \theta k D_0)v_j^{n+1} = (I + (1-\theta)kD_0)v_j^n \tag{1}$$

is unconditionally stable for $\theta \geq \frac{1}{2}$.

As we have done previously, we begin by considering simple wave solutions of the form

$$v_j^n = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n(\omega).$$

Substituting into Equation 1 yields

$$\begin{aligned}(I - \theta k D_0)e^{i\omega x_j}\hat{v}^{n+1}(\omega) &= (I + (1-\theta)kD_0)e^{i\omega x_j}\hat{v}^n(\omega)\\ \left(I - \frac{\theta\lambda}{2}(E - E^{-1})\right)e^{i\omega jh}\hat{v}^{n+1}(\omega) &= \left(I + \frac{(1-\theta)\lambda}{2}(E - E^{-1})\right)e^{i\omega jh}\hat{v}^n(\omega),\end{aligned}$$

where we have used the following identities: $D_0 = (E - E^{-1})/2h$, $\lambda = k/h$, and $x_j = jh$. Simplifying this equation, we obtain the following expressions

$$\begin{aligned}\left(e^{i\omega jh} - \frac{\theta\lambda}{2}(e^{i\omega(j+1)h} - e^{i\omega(j-1)h})\right)\hat{v}^{n+1}(\omega) &= \left(e^{i\omega jh} + \frac{(1-\theta)\lambda}{2}(e^{i\omega(j+1)h} - e^{i\omega(j-1)h})\right)\hat{v}^n(\omega)\\ \left(1 - \frac{\theta\lambda}{2}(e^{i\omega h} - e^{-i\omega h})\right)\hat{v}^{n+1}(\omega) &= \left(1 + \frac{(1-\theta)\lambda}{2}(e^{i\omega h} - e^{-i\omega h})\right)\hat{v}^n(\omega)\\ (1 - i\theta\lambda\sin\xi)\,\hat{v}^{n+1}(\omega) &= (1 + i(1-\theta)\lambda\sin\xi))\,\hat{v}^n(\omega), \qquad (2)\end{aligned}$$

where $\xi = \omega h$. From Equation 2 we find the following form of the symbol $\hat{Q}$.

$$\hat{v}^{n+1}(\omega) = \hat{Q}\hat{v}^n(\omega), \quad \hat{Q} = \frac{1 + i(1-\theta)\lambda\sin\xi}{1 - i\theta\lambda\sin\xi} \tag{3}$$

Recall from page 44 in [1] that we consider a method stable if

$$\sup_{0 \leq t_n \leq T, \omega, k, h} |\hat{Q}^n| \leq K(T),$$

as $h, k \to 0$. As was done in the textbook, we can choose σ, k, and h such that $|\hat{Q}| \leq 1 \Rightarrow |\hat{Q}|^2 \leq 1$. Also recall that for complex numbers z_1 and z_2, the squared modulus satisfies $|z_1|^2 = z_1\bar{z}_1$ and $|z_1/z_2| = |z_1|/|z_2|$. Applying these identities to the symbol $\hat{Q}$, we derive the following result.

$$\begin{aligned}|\hat{Q}|^2 &= \frac{|1 + i(1-\theta)\lambda\sin\xi|^2}{|1 - i\theta\lambda\sin\xi|^2}\\ &= \frac{1 + i(1-\theta)\lambda\sin\xi}{1 - i\theta\lambda\sin\xi} \cdot \frac{1 - i(1-\theta)\lambda\sin\xi}{1 + i\theta\lambda\sin\xi}\\ &= \frac{1 + (1-\theta)^2\lambda^2\sin^2\xi}{1 + \theta^2\lambda^2\sin^2\xi} \leq 1 \qquad (4)\end{aligned}$$

From Equation 4, we have

$$\begin{aligned}1+(1-\theta)^2\lambda^2\sin^2\xi &\leq 1+\theta^2\lambda^2\sin^2\xi\\(1-\theta)^2 &\leq \theta^2\\ \theta^2-2\theta+1 &\leq \theta^2, \quad \Rightarrow \theta \geq \frac{1}{2}.\end{aligned}$$

This proves that Equation 1 is unconditionally stable (i.e., stable for all values of λ) if and only if $\theta \geq \frac{1}{2}$. (QED)

Problem 2

When deriving the order of accuracy, Taylor expansion around some point (x_*, t_*) is used. Prove that (x_*, t_*) can be chosen arbitrarily and, in particular, that it does not have to be a gridpoint.

Let us begin by considering the discrete difference approximation to $u_t = u_x$ given by

$$v_j^{n+1} = (I + kD_0)v_j^n + \sigma k h D_+ D_- v_j^n.$$

Expanding the difference operators and collecting terms, we find the following expression.

$$\frac{v_j^{n+1}-v_j^n}{k} - \frac{v_{j+1}^n - v_{j-1}^n}{2h} - \sigma h\left(\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2}\right) = 0 \tag{5}$$

In order to estimate the truncation error, we follow the approach outlined in §2.4 from [1]. Specifically, we will calculate how well u satisfies the difference approximation in Equation 5. Assuming u is a smooth function, substitution into the previous expression yields

$$\tau_j^n \triangleq \frac{u_j^{n+1}-u_j^n}{k} - \frac{u_{j+1}^n - u_{j-1}^n}{2h} - \sigma h\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right),$$

where τ_j^n is defined as the truncation error evaluated at (x_j, t_n). This expression is equivalent to Equation 2.4.6 on page 60 in [1] and holds at all gridpoints (x_j, t_n). Note, however, that we can evaluate the truncation error $\tau(x_*, t_*)$ about any arbitrary point (x_*, t_*) to obtain

$$\begin{aligned}\tau(x_*, t_*) \triangleq & \frac{u(x_*, t_*+k) - u(x_*, t_*)}{k} - \\ & \frac{u(x_*+h, t_*) - u(x_*-h, t_*)}{2h} - \\ & \sigma h\left(\frac{u(x_*+h, t_*) - 2u(x_*, t_*) + u(x_*-h, t_*)}{h^2}\right)\end{aligned} \tag{6}$$

At this point, we recall that the following Taylor series expansions were derived on page 59 in [1] and are valid around any point (x, t).

$$\frac{u(x,t+k)-u(x,t)}{k} = u_t(x,t) + \frac{k}{2}u_{tt}(x,t) + \frac{k^2}{3!}\psi_0(x,t) \tag{7}$$

$$\frac{u(x+h,t)-u(x-h,t)}{2h} = u_x(x,t) + \frac{h^2}{3!}u_{xxx}(x,t) + \frac{h^4}{5!}\varphi_0(x,t) \tag{8}$$

$$\frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} = u_{xx}(x,t) + \frac{2h^2}{4!}u_{xxxx}(x,t) + \frac{2h^4}{6!}\varphi_1(x,t) \tag{9}$$

Substituting Equations 7, 8, and 9 into Equation 6, we obtain the following form for the truncation error about (x_*, t_*).

$$\tau(x_*, t_*) = u_t(x_*, t_*) - u_x(x_*, t_*) + \frac{k}{2} u_{tt}(x_*, t_*) - \sigma h u_{xx}(x_*, t_*) + \mathcal{O}(h^2 + k^2)$$

Since $u_t = u_x$, we can simplify this expression as follows.

$$\tau(x_*, t_*) = \left(\frac{k}{2} - \sigma h\right) u_{xx}(x_*, t_*) + \mathcal{O}(h^2 + k^2) \tag{10}$$

Recall that the truncation error is said to be accurate of order (p, q) if $\tau(x_*, t_*) = \mathcal{O}(h^p + k^q)$. In conclusion, we find that the scheme in Equation 5 is accurate of order $(1, 1)$ for $\sigma \neq k/(2h)$ and of order $(2, 2)$ otherwise. Note that this conclusion is valid for any point (x_*, t_*), yet it agrees with the result found at specific gridpoints (x_j, t_n) given in [1] on page 60.

In general, we could have considered any scheme (rather than the specific analysis that was presented for Equation 5). For any scheme, the Taylor series expansions derived on page 59 in [1] will hold for any point (x_*, t_*). As a result, the leading order behavior of the Taylor series expansion will be identical, resulting in the same truncation error $\tau(x_*, t_*)$ for all values of (x_*, t_*). (QED)

Problem 3

Prove that the leap-frog scheme

$$v_j^{n+1} = v_j^{n-1} + \lambda(v_{j+1}^n - v_{j-1}^n) \tag{11}$$

and the Crank-Nicholson scheme

$$\left(I - \frac{k}{2} D_0\right) v_j^{n+1} = \left(I + \frac{k}{2} D_0\right) v_j^n \tag{12}$$

are accurate of order $(2, 2)$. Despite the same order of accuracy, one can expect that one scheme is more accurate than the other. Why is that so?

Let's begin by rearranging terms in Equation 11 to obtain

$$\frac{v_j^{n+1} - v_j^{n-1}}{2k} = \frac{v_{j+1}^n - v_{j-1}^n}{2h},$$

where $\lambda = k/h$. Recall that the central difference satisfies

$$D_0 v_j^n = \frac{v_{j+1}^n - v_{j-1}^n}{2h}$$

and as a result, the previous expression reduces to

$$\frac{v_j^{n+1} - v_j^{n-1}}{2k} - D_0 v_j^n = 0. \tag{13}$$

In order to estimate the truncation error, we follow the approach outlined in §2.4 from [1]. Specifically, we will calculate how well u satisfies the difference approximation in Equation 13. Assuming u is a smooth function, substitution into the previous expression yields

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} - D_0 u_j^n = 0$$

$$\frac{u_j^{n+1}-u_j^{n-1}}{2k}-\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0. \tag{14}$$

Recall that the following Taylor series expansions about (x,t) are given on page 59 of [1]

$$D_0u(x,t)=\frac{u(x+h,t)-u(x-h,t)}{2h}=u_x(x,t)+\frac{h^2}{3!}u_{xxx}(x,t)+\frac{h^4}{5!}\varphi_0(x,t) \tag{15}$$

$$\frac{u(x,t+k)-u(x,t-k)}{2k}=u_t(x,t)+\frac{k^2}{3!}u_{ttt}(x,t)+\frac{k^4}{5!}\psi_1(x,t), \tag{16}$$

where

$$|\varphi_0(x,t)|\le\max_{x-h\le\xi\le x+h}\left|\frac{\partial^5u(\xi,t)}{\partial x^5}\right| \quad \textit{and} \quad |\psi_1(x,t)|\le\max_{t-k\le\xi\le t+k}\left|\frac{\partial^5u(x,\xi)}{\partial t^5}\right|.$$

Substituting Equations 15 and 16 into Equation 14, we obtain

$$u_t(x_j,t_n)-u_x(x_j,t_n)+\frac{k^2}{3!}u_{ttt}(x_j,t_n)-\frac{h^2}{3!}u_{xxx}(x_j,t_n)+\mathcal{O}(h^4+k^4)\triangleq\tau_j^n,$$

where τ_j^n is the *truncation error*. Recall that Equation 11 approximates the solution to the differential equation $u_t=u_x$. As a result, we have $u_{ttt}=u_{xxx}$ and the previous expression can be reduced to

$$\tau_j^n=\left(\frac{k^2}{3!}-\frac{h^2}{3!}\right)u_{xxx}(x_j,t_n)+\mathcal{O}(h^4+k^4)=\mathcal{O}(h^2+k^2) \tag{17}$$

Recall that the truncation error is said to be accurate of order (p,q) if $\tau=\mathcal{O}(h^p+k^q)$. As a result, **the leap-frog scheme in Equation 11 is accurate of order $(2,2)$, by Equation 17.** (QED)

Now let's consider the Crank-Nicholson scheme in Equation 12. Rearranging terms gives the following expression.

$$\frac{v_j^{n+1}-v_j^n}{k}-\frac{1}{2}D_0v_j^{n+1}-\frac{1}{2}D_0v_j^n=0$$

Assuming u is a smooth function, substitution into the previous expression yields

$$\frac{u_j^{n+1}-u_j^n}{k}-\frac{1}{2}D_0u_j^{n+1}-\frac{1}{2}D_0u_j^n=0. \tag{18}$$

Note that the Taylor series expansion, about (x,t), for the third term is given by Equation 15. Similarly, the expansion of the first term is given on page 59 in [1] as

$$\begin{aligned}\frac{u(x,t+k)-u(x,t)}{k}&=u_t(x,t)+\frac{k}{2}u_{tt}(x,t)+\frac{k^2}{3!}\psi_0(x,t)\\&=u_x(x,t)+\frac{k}{2}u_{xx}(x,t)+\frac{k^2}{3!}\psi_0(x,t),\end{aligned} \tag{19}$$

since $u_t=u_x$ and where

$$|\psi_0(x,t)|\le\max_{t\le\xi\le t+k}\left|\frac{\partial^3u(x,\xi)}{\partial t^3}\right|.$$

To complete the derivation, we must find the Taylor series expansion for the second term in Equation 18. By analogy to Equation 15, the expansion about $(x,t+k)$ is given by

$$\begin{aligned}D_0u(x,t+k)&=\frac{u(x+h,t+k)-u(x-h,t+k)}{2h}\\&=u_x(x,t+k)+\frac{h^2}{3!}u_{xxx}(x,t+k)+\frac{h^4}{5!}\varphi_0(x,t+k).\end{aligned}$$

Note that the Taylor series expansion about (x, t) for the leading term is given by

$$\begin{aligned} u_x(x, t+k) &= u_x(x,t) + k u_{xt}(x,t) + \frac{k^2}{2} u_{xtt}(x,t) + \mathcal{O}(k^3) \\ &= u_x(x,t) + k u_{xx}(x,t) + \frac{k^2}{2} u_{xxx}(x,t) + \mathcal{O}(k^3), \end{aligned}$$

since $u_t = u_x$. Substituting this result into the previous expression, we have

$$D_0 u(x, t+k) = u_x(x,t) + k u_{xx}(x,t) + \left(\frac{k^2}{2} + \frac{h^2}{3!}\right) u_{xxx}(x,t) + \ldots, \tag{20}$$

where higher-order terms have been omitted. Substituting Equations 15, 19, and 20 into Equation 18 gives the following result (when considering only the leading-order terms in h and k).

$$u_x(x,t) + \frac{k}{2} u_{xx}(x,t) - \frac{1}{2} u_x(x,t) - \frac{k}{2} u_{xx}(x,t) - \left(\frac{k^2}{4} + \frac{h^2}{2 \cdot 3!}\right) u_{xxx}(x,t) - \frac{1}{2} u_x(x,t) - \frac{h^2}{2 \cdot 3!} u_{xxx}(x,t) = \tau$$

Simplifying this expression yields the following form for the truncation error.

$$\tau = -\left(\frac{k^2}{4} + \frac{h^2}{3!}\right) u_{xxx}(x,t) = \mathcal{O}(h^2 + k^2) \tag{21}$$

In conclusion, **the Crank-Nicholson scheme in Equation 12 is accurate of order $(2, 2)$, by Equation 21.** (QED)

Although both the leap-frog and Crank-Nicholson schemes have the same order of accuracy, **one can expect that the leap-frog scheme is more accurate**. From Equations 17 and 21, we know that the schemes have the following leading-order truncation errors.

$$\begin{aligned} \tau_{\text{leap-frog}} &= +\left(\frac{k^2}{3!} - \frac{h^2}{3!}\right) u_{xxx}(x,t) \\ \tau_{\text{Crank-Nicholson}} &= -\left(\frac{k^2}{4} + \frac{h^2}{3!}\right) u_{xxx}(x,t) \end{aligned}$$

Note that both schemes have identical "spatial" truncation errors of $-\frac{h^2}{3!} u_{xxx}(x,t)$, however the magnitude of the "temporal" error for the leap-frog method is smaller than that for the Crank-Nicholson scheme (i.e., $k^2/3! < k^2/4$). As a result, for a fixed value of $\lambda = k/h$ for which both schemes are stable, one would expect that the leap-frog method would predict the solution with $\approx$66.7% of the L_2-error exhibited by the Crank-Nicholson scheme. This result demonstrates that the order of accuracy does not fully specify the truncation errors and, in certain situations, the coefficients of the leading-order terms will decide which scheme is more accurate.

References

[1] Bertil Gustafsson, Heinz-Otto Kreiss, and Joseph Oliger. *Time Dependent Problems and Difference Methods*. John Wiley & Sons, 1995.