AM 255: Problem Set 4

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Problem 1

Prove that the θ scheme

$$(I - \theta k D_+ D_-) v_j^{n+1} = (I + (1-\theta) k D_+ D_-) v_j^n, \; j = 0, 1, \ldots, N, \; 0 \le \theta \le 1 \tag{1}$$

is unconditionally stable for $\theta \ge \frac{1}{2}$.

In order to find the necessary conditions for stability, we begin by making the ansatz

$$v_j^n = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n(\omega),$$

where the solution is composed of a single Fourier component. Substituting this expression into Equation 1, we obtain the following result.

$$\frac{1}{\sqrt{2\pi}} \hat{v}^{n+1}(\omega) \left(I - \theta k D_+ D_-\right) e^{i\omega x_j} = \frac{1}{\sqrt{2\pi}} \hat{v}^n(\omega) \left(I + (1-\theta) k D_+ D_-\right) e^{i\omega x_j}$$

$$\Rightarrow \hat{v}^{n+1}(\omega) \left(e^{i\omega x_j} - \theta k D_+ D_- e^{i\omega x_j}\right) = \hat{v}^n(\omega) \left(e^{i\omega x_j} + (1-\theta) k D_+ D_- e^{i\omega x_j}\right) \tag{2}$$

In order to proceed, we require the following identity (given by Equation 2.5.7 in [1]).

$$k D_+ D_- e^{i\omega x_j} = -4\sigma \sin^2 \frac{\xi}{2} e^{i\omega x_j}, \text{ where } \sigma = \frac{k}{h^2} \text{ and } \xi = \omega h \tag{3}$$

Substituting Equation 3 into Equation 2 gives the following expressions.

$$\hat{v}^{n+1}(\omega) \left(e^{i\omega x_j} + 4\theta\sigma \sin^2 \frac{\xi}{2} e^{i\omega x_j}\right) = \hat{v}^n(\omega) \left(e^{i\omega x_j} + 4(\theta - 1)\sigma \sin^2 \frac{\xi}{2} e^{i\omega x_j}\right)$$

$$\Rightarrow \hat{v}^{n+1}(\omega) \left(1 + 4\theta\sigma \sin^2 \frac{\xi}{2}\right) = \hat{v}^n(\omega) \left(1 + 4(\theta - 1)\sigma \sin^2 \frac{\xi}{2}\right)$$

Simplifying this equation gives a closed-form expression for the symbol $\hat{Q}$.

$$\hat{v}^{n+1}(\omega) = \hat{Q}\hat{v}^n(\omega), \quad \hat{Q} = \frac{1 + 4(\theta-1)\sigma \sin^2 \frac{\xi}{2}}{1 + 4\theta\sigma \sin^2 \frac{\xi}{2}} \tag{4}$$

Recall from page 44 in [1] that we consider a method stable if

$$\sup_{0 \le t_n \le T, \omega, k, h} |\hat{Q}^n| \le K(T),$$

as $h, k \to 0$. As was done in the textbook, we can choose σ, k, and h such that

$$|\hat{Q}| \le 1.$$

Substituting the expression for the symbol $\hat{Q}$ from Equation 4, we derive the following equation.

$$|\hat{Q}| = \left| \frac{1 + 4(\theta - 1)\sigma \sin^2 \frac{\xi}{2}}{1 + 4\theta\sigma \sin^2 \frac{\xi}{2}} \right| \leq 1$$

Recall that, for complex numbers z_1 and z_2, the modulus satisfies: $|z_1 + z_2| \leq |z_1| + |z_2|$, $|z_1 z_2| = |z_1||z_2|$, and $|z_1/z_2| = |z_1|/|z_2|$. Applying these identities to the previous expression, we derive the following result.

$$\left| \frac{1 + 4(\theta - 1)\sigma \sin^2 \frac{\xi}{2}}{1 + 4\theta\sigma \sin^2 \frac{\xi}{2}} \right| = \frac{\left|1 + 4(\theta - 1)\sigma \sin^2 \frac{\xi}{2}\right|}{\left|1 + 4\theta\sigma \sin^2 \frac{\xi}{2}\right|} \leq 1$$

Multiplying by the denominator gives

$$\left|1 + 4(\theta - 1)\sigma \sin^2 \frac{\xi}{2}\right| \leq \left|1 + 4\theta\sigma \sin^2 \frac{\xi}{2}\right|.$$

Further simplifying, we can eliminate the dependence on σ and ξ as follows.

$$|1| + 4\,|\theta - 1| \left|\sigma \sin^2 \frac{\xi}{2}\right| \leq |1| + 4\,|\theta| \left|\sigma \sin^2 \frac{\xi}{2}\right|$$

$$\Rightarrow |\theta - 1| \leq |\theta|$$

Note that, for $0 \leq \theta \leq 1$, we must have $|\theta - 1| = 1 - \theta$ and $|\theta| = \theta$. Substituting these identities into the previous expression gives the desired result.

$$1 - \theta \leq \theta \quad \Rightarrow \quad \boxed{\theta \geq \frac{1}{2}}$$

In conclusion, this proves that Equation 1 is unconditionally stable (i.e., stable for all values of λ) if and only if $\theta \geq \frac{1}{2}$. (QED)

Problem 2

Prove that the truncation errors for the backward Euler and Crank-Nicholson schemes, applied to $u_t = u_{xx}$, are $\mathcal{O}(h^2 + k)$ and $\mathcal{O}(h^2 + k^2)$, respectively. Despite having the same order of approximation, explain why at certain times the backward Euler method is more accurate for the example in §2.5 in [1].

Recall (from Equation 2.5.17 in [1]) that the backward Euler scheme for $u_t = u_{xx}$ is given by

$$(I - kD_+D_-)v_j^{n+1} = v_j^n, \; j = 0, 1, \ldots, N. \tag{5}$$

Rearranging terms, we obtain the following expression.

$$\frac{v_j^{n+1} - v_j^n}{k} - D_+D_-v_j^{n+1} = 0 \tag{6}$$

In order to estimate the truncation error, we follow the approach outlined in §2.4 from [1]. Specifically, we will calculate how well u satisfies the difference approximation in Equation 6. Assuming u is a smooth function, substitution into the previous expression yields the following form for the truncation error τ_j^n.

$$\tau_j^n = \frac{u_j^{n+1} - u_j^n}{k} - D_+D_-u_j^{n+1}$$

$$\Rightarrow \tau(x_j, t_n) = \frac{u(x_j, t_n + k) - u(x_j, t_n)}{k} - D_+D_-u(x_j, t_n + k) \tag{7}$$

Recall that the following Taylor series expansions about (x, t) are given on page 59 of [1]

$$\begin{aligned} D_+D_-u(x,t) &= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \\ &= u_{xx}(x,t) + \frac{2h^2}{4!}u_{xxxx}(x,t) + \frac{2h^4}{6!}\varphi_1(x,t) \end{aligned} \tag{8}$$

$$\frac{u(x,t+k) - u(x,t)}{k} = u_t(x,t) + \frac{k}{2}u_{tt}(x,t) + \frac{k^2}{3!}u_{ttt}(x,t) + \frac{k^4}{4!}\psi_0(x,t), \tag{9}$$

where

$$|\varphi_1(x,t)| \le \max_{x-h\le\xi\le x+h} |\frac{\partial^6 u(\xi,t)}{\partial x^6}| \quad \text{and} \quad |\psi_0(x,t)| \le \max_{t\le\xi\le t+k} |\frac{\partial^4 u(x,\xi)}{\partial t^4}|.$$

Note that we must modify Equation 8 such that it represents the correct expansion about $(x, t + k)$; this can be achieved by applying the following Taylor series expansions.

$$\begin{aligned} u_{xx}(x,t+k) &= u_{xx}(x,t) + ku_{xxt}(x,t) + \frac{k^2}{2}u_{xxtt}(x,t) + \ldots \\ &= u_t(x,t) + ku_{tt}(x,t) + \frac{k^2}{2}u_{ttt}(x,t) + \ldots \end{aligned} \tag{10}$$

$$\begin{aligned} u_{xxxx}(x,t+k) &= u_{xxxx}(x,t) + ku_{xxxxt}(x,t) + \ldots \\ &= u_{tt}(x,t) + ku_{ttt}(x,t) + \ldots \end{aligned} \tag{11}$$

Note that in the previous equations we have applied the condition $u_t = u_{xx}$. Substituting Equations 10 and 11 into Equation 8 gives the following Taylor series expansion.

$$D_+D_-u(x,t+k) = u_t(x,t) + \left(\frac{2h^2}{4!} + k\right) u_{tt}(x,t) + \left(\frac{2h^4}{6!} + \frac{k^2}{2} + \frac{2h^2k}{4!}\right) u_{ttt}(x,t) + \ldots \quad (12)$$

At this point, we can substitute Equations 9 and 12 into Equation 7 to estimate the truncation error.

$$\begin{aligned}\tau(x_j,t_n) &= u_t(x_j,t_n) + \frac{k}{2}u_{tt}(x_j,t_n) - u_t(x_j,t_n) + \left(\frac{2h^2}{4!} - k\right) u_{tt}(x_j,t_n) + \ldots \\ &= -\left(\frac{2h^2}{4!} + \frac{k}{2}\right) u_{tt}(x_j,t_n) + \ldots = \mathcal{O}(h^2 + k) \qquad (13)\end{aligned}$$

Recall that the truncation error is said to be accurate of order (p,q) if $\tau = \mathcal{O}(h^p + k^q)$. As a result, **the backward Euler scheme in Equation 5 is accurate of order $(2,1)$ by Equation 13.** (QED)

Now let's consider the Crank-Nicholson scheme (as defined by Equation 2.5.19 in [1]).

$$\left(I - \frac{k}{2}D_+D_-\right) v_j^{n+1} = \left(I + \frac{k}{2}D_+D_-\right) v_j^n,\ j = 0,1,\ldots,N. \quad (14)$$

Rearranging terms, we obtain the following expression.

$$\frac{v_j^{n+1} - v_j^n}{k} - \frac{1}{2}D_+D_-v_j^{n+1} - \frac{1}{2}D_+D_-v_j^n = 0 \quad (15)$$

Assuming u is a smooth function, substitution into this expression yields the following form for the truncation error τ_j^n.

$$\begin{aligned}\tau_j^n &= \frac{u_j^{n+1} - u_j^n}{k} - \frac{1}{2}D_+D_-u_j^{n+1} - \frac{1}{2}D_+D_-u_j^n = 0 \\ \Rightarrow \tau(x_j,t_n) &= \frac{u(x_j,t_n+k) - u(x_j,t_n)}{k} - \frac{1}{2}D_+D_-u(x_j,t_n+k) - \frac{1}{2}D_+D_-u(x_j,t_n) \qquad (16)\end{aligned}$$

Conveniently, we have already derived all of the necessary Taylor expansions. Substituting Equations 8, 9, and 12 into Equation 16 gives the following estimate of the truncation error.

$$\begin{aligned}\tau(x_j,t_n) = \quad & \left\{u_t(x_j,t_n) + \frac{k}{2}u_{tt}(x_j,t_n) + \frac{k^2}{3!}u_{ttt}(x_j,t_n)\right\} - \\ & \frac{1}{2}\left\{u_t(x_j,t_n) + \left(\frac{2h^2}{4!} + k\right) u_{tt}(x_j,t_n) + \left(\frac{2h^4}{6!} + \frac{k^2}{2} + \frac{2h^2k}{4!}\right) u_{ttt}(x_j,t_n)\right\} - \\ & \frac{1}{2}\left\{u_t(x_j,t_n) + \frac{2h^2}{4!}u_{tt}(x_j,t_n) + \frac{2h^4}{6!}u_{ttt}(x_j,t_n)\right\} + \ldots\end{aligned}$$

Simplifying, we obtain the following expression.

$$\tau(x_j,t_n) = -\frac{2h^2}{4!}u_{tt}(x_j,t_n) - \left(\frac{k^2}{12} + \frac{2h^4}{6!} + \frac{h^2k}{4!}\right) u_{ttt}(x_j,t_n) = \mathcal{O}(h^2 + k^2) \quad (17)$$

In conclusion, **the Crank-Nicholson scheme in Equation 14 is accurate of order $(\mathbf{2}, \mathbf{2})$, by Equation 17.** (QED)

As discussed on pages 67-69 in [1], both the backward Euler and Crank-Nicholson schemes are unconditionally stable, with symbols given by

$$\hat{Q}_{backwards\ Euler} = \frac{1}{1 + 4\sigma \sin^2 \frac{\xi}{2}} \quad \text{and} \quad \hat{Q}_{Crank\text{-}Nicholson} = \frac{1 - 2\sigma \sin^2 \frac{\xi}{2}}{1 + 2\sigma \sin^2 \frac{\xi}{2}}$$

for $\sigma = k/h^2$. In general, we would like to use time steps of the same order as the space step; in this case, $\sigma = \mathcal{O}(1/h)$ and $\hat{Q}_{Crank\text{-}Nicholson} \to -1$. Correspondingly, we find that there is very little damping for the Crank-Nicholson scheme in this situation (i.e., $|\hat{Q}_{Crank\text{-}Nicholson}| \to 1$). As a result, we expect that backward Euler scheme will have better numerical stability for all possible values of σ.

Problem 3

Consider the convection-diffusion equation given on page 70 in [1].

$$u_t + au_x = \eta u_{xx}, \quad -\infty < x < \infty,\ 0 \le t$$
$$u(x,0) = \sin(x), \quad -\infty < x < \infty$$

Compute the discrete difference approximation at time $T = 1$ given by

$$v_j^{n+1} = v_j^n + k\left(\eta D_+ D_- - aD_0\right) v_j^n, \ j = 0, 1, \ldots, N, \tag{18}$$

where $a = \eta = 1$. Consider discrete grids of size $N = \{10, 20, 40, 80, 160, 320\}$. Evaluate the time step k consistent with

$$\alpha = \frac{2\eta k}{h^2}, \ \alpha \le 1$$

and also the time step k consistent with

$$\lambda = \frac{ak}{h}, \ |\lambda| \le 1.$$

Graphically compare the exact solution to the numerical solutions and tabulate the L_2-errors. Finally, estimate the order of approximation achieved for each value of the time step.

My implementation of the discrete difference approximation, as defined by Equation 18, was completed using Matlab and is included as `prob3.m`. Before presenting the results of my program, I will briefly outline the architecture of the source code. On lines 19-56 I select the values of $\{N, h, k\}$ and determine the resulting grid points $\{x, t\}$. (Note that on lines 47-49 I ensure that the last time is given by $T = 1$.) Lines 58-68 implement Equation 18. Note that I have implemented the difference operators D_0 and D_+D_- with stand-alone programs `D0.m` and `DpDm.m`, respectively. Finally, lines 70-107 create the tables and plots shown in this write-up.

To complete our analysis we require a closed-form solution for $u(x,t)$ at time $T = 1$ with $a = \eta = 1$. Recall (from Equation 2.6.3 in [1]) that the Fourier-space solution to the convection-diffusion equation is given by

$$\hat{u}(\omega, t) = e^{-(ia\omega + \eta\omega^2)t} \hat{f}(\omega) \quad \Rightarrow \quad \hat{u}(\omega, 1) = e^{-(i\omega + \omega^2)} \hat{f}(\omega), \text{ for } a = \eta = t = 1.$$

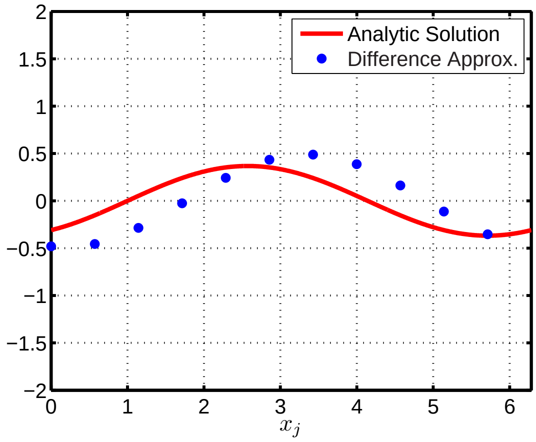
By the superposition principle, the solution for $u(x,1)$ can be written as

$$u(x,1) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} e^{i\omega x} e^{-(i\omega+\omega^2)} \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} e^{-\omega^2} e^{i\omega(x-1)} \hat{f}(\omega). \tag{19}$$

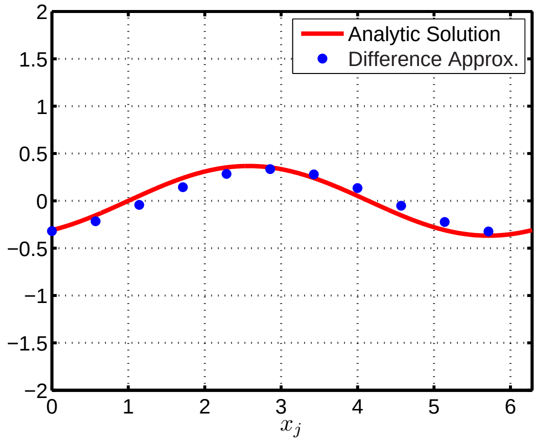
Note that the Fourier coefficients $\hat{f}(\omega)$ for $f(x) = \sin(x)$ can be obtained by inspection.

$$f(x) = \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)$$

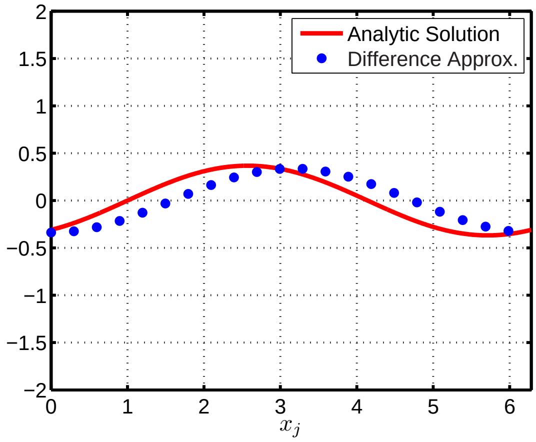
$$\Rightarrow \hat{f}(\omega) = \begin{cases} -i\sqrt{\frac{\pi}{2}} & \text{if } \omega = 1 \\ i\sqrt{\frac{\pi}{2}} & \text{if } \omega = -1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$



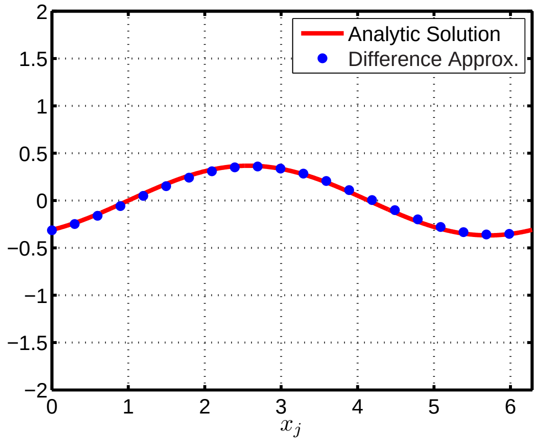
(a) $k = h$ and $N = 10$



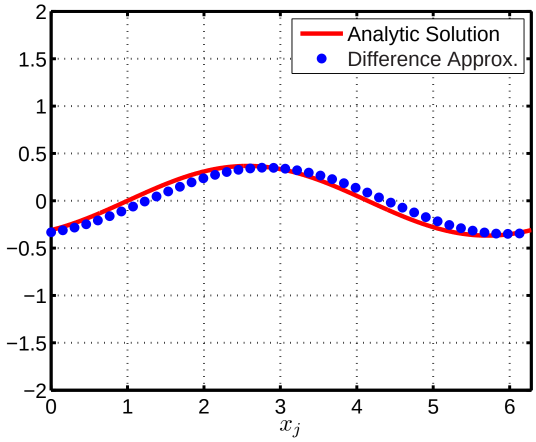
(b) $k = h^2/2$ and $N = 10$



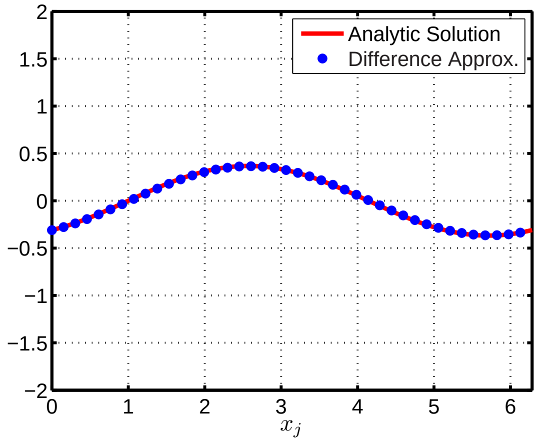
(c) $k = h$ and $N = 20$



(d) $k = h^2/2$ and $N = 20$



(e) $k = h$ and $N = 40$



(f) $k = h^2/2$ and $N = 40$

Figure 1: Comparison between difference approximations and the analytic solution of the convection-diffusion equation at time $T = 1$. The left column shows the results for the time step $k = h$, whereas the right column shows the results for the time step $k = h^2/2$.

Substituting Equation 20 into Equation 19 gives the desired analytic solution at $T = 1$.

$$u(x,1) = e^{-1}\left(\frac{e^{i(x-1)} - e^{-i(x-1)}}{2i}\right) \quad \Rightarrow \quad \boxed{u(x,1) = e^{-1}\sin(x-1)}$$

Given the analytic solution, all that remains to be done is to select the appropriate time steps. As required by the problem statement, we use the following two values for the time step k.

$$k_1 = \frac{\alpha h^2}{2\eta} \quad \text{and} \quad k_2 = \frac{\lambda h}{a}$$

Specifically, we choose $\alpha = \lambda = 1$ such that

$$k_1 = \frac{h^2}{2} \quad \text{and} \quad k_2 = h.$$

The approximation results for both $k = h$ and $k = h^2/2$ are tabulated below. In addition, the corresponding plots for $T = 1$ are included in Figure 1.

N	L_2-error	order
10	0.602	NA
20	0.366	$h^{0.72}$
40	0.161	$h^{1.19}$
80	4.004e5	$h^{-21.25}$
160	7.659e35	$h^{-100.59}$
319	7.620e103	$h^{-225.88}$

Table 3.1: $k = h$

N	L_2-error	order
10	0.178	NA
20	4.133e-2	$h^{2.10}$
40	1.262e-2	$h^{1.71}$
80	2.666e-3	$h^{2.24}$
160	8.108e-4	$h^{1.72}$
319	2.131e-4	$h^{1.93}$

Table 3.2: $k = h^2/2$

Note that the standard definition of the discrete L_2 norm was used to evaluate the total error as

$$L_2\text{-error}(N) \triangleq \sqrt{\sum_{j=0}^{N} |u(x_j, t^n) - v_j^n|^2 h}.$$

In addition, the following definition of order of approximation was given in class.

$$\text{order} \triangleq \log_2\left(\frac{L_2\text{-error}(N)}{L_2\text{-error}(2N)}\right)$$

In conclusion, we find that only $k = h^2/2$ results in a stable solution for large N. As shown in Table 3.1, when the number of points $N \geq 80$, the numerical solution with time step $k = h$ is unstable. Alternatively, we find the the numerical solution with time step $k = h^2/2$ is accurate to second-order for all values of N considered in this test. As a result, we can conclude that the discrete difference approximation in Equation 18 is stable for $k = h^2/2$. Note that, despite the fact that Equation 18 can achieve stability, the scheme remains undesirable as it requires too many time steps to compute a robust estimate in practical situations.

Problem 4

What explicit method could be used for the Schrödinger type equation

$$u_t = iu_{xx}?$$

Derive the stability condition.

From an examination of a variety of schemes, we found that the backwards Euler method was both simple and unconditionally stable for the Schrödinger type equation $u_t = iu_{xx}$. To begin our analysis, recall (from Equation 2.5.17 in [1]) that the backwards Euler scheme for the heat equation $u_t = u_{xx}$ is given by

$$(I - kD_+D_-)v_j^{n+1} = v_j^n,\ j = 0, 1, \ldots, N.$$

By inspection, we can modify this expression to model the Schrödinger type equation $u_t = iu_{xx}$ such that

$$\boxed{(I - ikD_+D_-)v_j^{n+1} = v_j^n,\ j = 0, 1, \ldots, N.} \tag{21}$$

Rearranging terms in this expression gives the following form for the approximation scheme.

$$\frac{v_j^{n+1} - v_j^n}{k} = iD_+D_-v_j^{n+1}$$

Clearly, the first term in this expression approximates u_t using the backward difference in time. Similarly, the second term approximates iu_{xx} using the "natural" centered difference operator D_+D_- applied to v_j^{n+1} and scaled by i. As a result, we verify that Equation 21 represents a valid backward Euler approximation for the Schrödinger type equation.

In order to analyze the stability of the proposed scheme, we make the typical ansatz

$$v_j^n = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n(\omega),$$

where the solution is composed of a single Fourier component. Substituting this expression into Equation 21, we obtain the following result.

$$\hat{v}^{n+1}(\omega)(e^{i\omega x_j} - ikD_+D_-e^{i\omega x_j}) = \hat{v}^n(\omega)$$

Recall (from Equations 2.7.7 and 2.7.8 in [1]) the following expression holds for $\xi = \omega h$.

$$D_+D_-e^{i\omega x_j} = \left(-\frac{4}{h^2}\sin^2\frac{\xi}{2}\right) e^{i\omega x_j} \tag{22}$$

Substituting this identity into the previous equation gives the following solution for the symbol $\hat{Q}$ (with $\sigma = k/h^2$).

$$\hat{v}^{n+1}(\omega) = \hat{Q}\hat{v}^n(\omega), \quad \hat{Q} = \frac{1}{1 + i4\sigma\sin^2\frac{\xi}{2}} \tag{23}$$

Recall from Problem 1 that we consider a method stable if

$$\sup_{0 \le t_n \le T, \omega, k, h} |\hat{Q}^n| \le K(T),$$

as $h, k \to 0$. As was done in previously, we can choose σ, k, and h such that

$$|\hat{Q}| \leq 1 \Rightarrow |\hat{Q}|^2 \leq 1.$$

Substituting the expression for the symbol $\hat{Q}$ from Equation 23, we derive the following inequalities.

$$|\hat{Q}|^2 = \left|\frac{1}{1 + i4\sigma \sin^2 \frac{\xi}{2}}\right|^2 \leq 1 \quad \Rightarrow \quad \left|1 + i4\sigma \sin^2 \frac{\xi}{2}\right|^2 \geq 1$$

Following the method presented on page 45 in [1], we can rewrite this expression as follows.

$$\left|1 + i4\sigma \sin^2 \frac{\xi}{2}\right|^2 = 1 + \left(4\sigma \sin^2 \frac{\xi}{2}\right)^2 \geq 1 \quad \Rightarrow \quad 16\sigma^2 \sin^4 \frac{\xi}{2} \geq 0$$

Simplifying this expression gives the following constraint on $\sigma = k/h^2$.

$$\sigma \geq 0$$

Since this expression holds of all λ, then we can conclude that the proposed backward Euler scheme in Equation 21 is **unconditionally stable**.

Problem 5

Define the Crank-Nicholson approximation for the Korteweg de Vries type equation

$$u_t = u_{xxx} + au_x.$$

Prove unconditional stability.

Recall (from Equation 2.3.3 in [1]) that the Crank-Nicholson scheme for $u_t = u_x$ is given by

$$\left(I - \frac{k}{2}D_0\right)v_j^{n+1} = \left(I + \frac{k}{2}D_0\right)v_j^n,\ j = 0, 1, \ldots, N. \tag{24}$$

Similarly, recall (from Equation 2.5.19 in [1]) that the Crank-Nicholson scheme for $u_t = u_{xx}$ is given by

$$\left(I - \frac{k}{2}D_+D_-\right)v_j^{n+1} = \left(I + \frac{k}{2}D_+D_-\right)v_j^n,\ j = 0, 1, \ldots, N. \tag{25}$$

Finally, note that (according to Equation 2.7.7 in [1]) the most natural centered difference approximation to the third partial derivative is given by

$$\frac{\partial^3}{\partial x^3} \to Q_3 = D_0(D_+D_-). \tag{26}$$

Combining Equations 24, 25, and 26, it is apparent that the corresponding Crank-Nicholson scheme for $u_t = u_{xxx} + au_x$ is given by

$$\boxed{\left(I - \frac{k}{2}\left(aD_0 + D_0D_+D_-\right)\right)v_j^{n+1} = \left(I + \frac{k}{2}\left(aD_0 + D_0D_+D_-\right)\right)v_j^n,\ j = 0, 1, \ldots, N.} \tag{27}$$

In order to analyze the stability condition for this scheme, we made the typical ansatz

$$v_j^n = \frac{1}{\sqrt{2\pi}}e^{i\omega x_j}\hat{v}^n(\omega),$$

where the solution is composed of a single Fourier component. Substituting this expression into Equation 27, we obtain the following result.

$$\hat{v}^{n+1}(\omega)\left(1 - \frac{k}{2}\left(aD_0 + D_0D_+D_-\right)\right)e^{i\omega x_j} = \hat{v}^n(\omega)\left(1 + \frac{k}{2}\left(aD_0 + D_0D_+D_-\right)\right)e^{i\omega x_j} \tag{28}$$

Recall (from Equations 1.2.3 and 2.7.8 in [1]) the following expressions hold for $\xi = \omega h$.

$$D_0e^{i\omega x_j} = \frac{i}{h}\sin(\xi)e^{i\omega x_j} \tag{29}$$

$$D_0D_+D_-e^{i\omega x_j} = \frac{i}{h}\sin(\xi)\left(-\frac{4}{h^2}\sin^2\frac{\xi}{2}\right)e^{i\omega x_j} \tag{30}$$

Substituting Equations 29 and 30 into Equation 28 and canceling the common terms in $e^{i\omega x_j}$ gives the following expression in $\lambda = k/h$.

$$\hat{v}^{n+1}(\omega)\left\{1+i\left(\frac{a\lambda}{2}\right)\sin(\xi)-2i\lambda\sin(\xi)\left(\frac{\sin^2(\xi)}{h^2}\right)\right\}=$$
$$\hat{v}^{n}(\omega)\left\{1-i\left(\frac{a\lambda}{2}\right)\sin(\xi)+2i\lambda\sin(\xi)\left(\frac{\sin^2(\xi)}{h^2}\right)\right\}$$

Simplifying this expression gives a closed-form solution for the symbol $\hat{Q}$.

$$\hat{v}^{n+1}(\omega)=\hat{Q}\hat{v}^{n}(\omega), \quad \hat{Q}=\frac{1-i\lambda\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)}{1+i\lambda\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)} \tag{31}$$

Recall from Problem 1 that we consider a method stable if

$$\sup_{0\le t_n\le T,\omega,k,h}|\hat{Q}^n|\le K(T),$$

as $h,k\to 0$. As was done in previously, we can choose σ, k, and h such that

$$|\hat{Q}|\le 1.$$

Substituting the expression for the symbol $\hat{Q}$ from Equation 31, we derive the following equation.

$$|\hat{Q}|=\left|\frac{1-i\lambda\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)}{1+i\lambda\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)}\right|\le 1$$

Recall that, for complex numbers z_1 and z_2, the modulus satisfies: $|z_1+z_2|\le|z_1|+|z_2|$, $|z_1z_2|=|z_1||z_2|$, and $|z_1/z_2|=|z_1|/|z_2|$. Applying these identities to the previous expression, we derive the following result.

$$\left|1-i\lambda\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)\right|\le\left|1+i\lambda\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)\right|$$

Simplifying further, we obtain the following expressions.

$$|1|+|\lambda|\left|\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)\right|\le|1|+|\lambda|\left|\left(\frac{a}{2}\sin\xi-\frac{2}{h^2}\sin\xi\sin^2\frac{\xi}{2}\right)\right|$$
$$\Rightarrow|\lambda|\le|\lambda|$$

Since this expression holds of all λ, then we can conclude that the Crank-Nicholson scheme in Equation 27 is unconditionally stable. (QED)

Problem 6

Derive the stability condition for the Euler approximation to $u_t = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$. Prove the DuFort-Frankel method is unconditionally stable for the same equation.

Recall (from Equation 2.5.6 in [1]) that the Euler approximation to the one-dimensional heat equation $u_t = u_{xx}$ is given by

$$v_j^{n+1} = (I + kD_+D_-)v_j^n,\ j = 0, 1, \ldots, N. \tag{32}$$

Following the approach outlined in §2.8 in [1], we can extend this equation to three space dimensions as follows.

$$v_j^{n+1} = (I + k(D_{+x_1}D_{-x_1} + D_{+x_2}D_{-x_2} + D_{+x_3}D_{-x_3}))v_j^n,\ j = 0, 1, \ldots, N. \tag{33}$$

Recall (from Equation 2.5.8 in [1]) that the transformed difference scheme in Equation 32 is given by

$$\hat{v}^{n+1}(\omega) = \hat{Q}\hat{v}^n(\omega), \quad \hat{Q} = 1 - 4\sigma \sin^2 \frac{\xi}{2},$$

where $\xi = \omega h$ and $\sigma = k/h^2$. Once again, we can extend this expression to represent the equivalent tranformed difference scheme in Equation 33 such that

$$\hat{v}^{n+1}(\omega) = \hat{Q}\hat{v}^n(\omega), \quad \hat{Q} = 1 - 4\sigma \left(\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + \sin^2 \frac{\xi_3}{2}\right). \tag{34}$$

Recall from Problems 1 and 5 that we consider a method stable if

$$\sup_{0 \le t_n \le T, \omega, k, h} |\hat{Q}^n| \le K(T),$$

as $h, k \to 0$. As was done several times previously, we can choose σ, k, and h such that

$$|\hat{Q}| \le 1.$$

Substituting the expression for the symbol $\hat{Q}$ from Equation 34, we derive the following equation.

$$|\hat{Q}| = \left|1 - 4\sigma \left(\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + \sin^2 \frac{\xi_3}{2}\right)\right| \le 1$$
$$\Rightarrow -1 \le 1 - 4\sigma \left(\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + \sin^2 \frac{\xi_3}{2}\right) \le 1$$

Simplifying this expression gives the following result.

$$0 \le 2\sigma \left(\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + \sin^2 \frac{\xi_3}{2}\right) \le 1$$

Note that $\left(\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + \sin^2 \frac{\xi_3}{2}\right) \le 3$; substituting this upper bound in the previous expression yields the stability criterion for the Euler approximation given by Equation 33.

$$\therefore \boxed{\sigma \le \frac{1}{6}}$$

We now turn our attention to proving that the DuFort-Frankel method is unconditionally stable for $u_t = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$. Recall (from Equation 2.5.12 in [1]) that the DuFort-Frankel approximation to the one-dimensional heat equation $u_t = u_{xx}$ is given by

$$v_j^{n+1} = 2\sigma(v_{j+1}^n - v_j^{n+1} - v_j^{n-1} + v_{j-1}^n) + v_j^{n-1},\ j = 0, 1, \ldots, N, \tag{35}$$

for $\sigma = k/h^2$. Note that we can rewrite the previous expression as follows.

$$v_j^{n+1} = \left(\frac{2\sigma}{1+2\sigma}\right)(v_{j+1}^n + v_{j-1}^n) + \left(\frac{1-2\sigma}{1+2\sigma}\right)v_j^{n-1}$$

As done on page 66 in [1], we can make the ansatz $\hat{v}^n(\omega) = z^n$ to obtain the following *characteristic equation* with $z \in \mathbb{C}$ and $\xi = \omega h$.

$$z^2 - \left(\frac{4\sigma}{1+2\sigma}\right)(cos\xi)z - \left(\frac{1-2\sigma}{1+2\sigma}\right) = 0$$

Applying the quadratic formula, we obtain the following two solutions for z given by

$$z_{1,2} = \left(\frac{2\sigma}{1+2\sigma}\right)(cos\xi) \pm \left(\frac{1}{1+2\sigma}\right)\sqrt{A},$$

where $A = 4\sigma^2\cos^2\xi + 1 - 4\sigma^2$. As shown on page 66 in [1] we can prove that $|z_{1,2}| \leq 1$ as follows. First, if $A \geq 0$, then $A \leq 1$ and

$$|z_{1,2}| \leq \frac{2\sigma}{1+2\sigma} + \frac{1}{1+2\sigma} = 1 \quad \Rightarrow \quad |z_{1,2}| \leq 1.$$

Similarly, if $A < 0$, then we have

$$|z_{1,2}|^2 = \frac{4\sigma^2 - 1}{(1+2\sigma)^2} = \frac{2\sigma - 1}{2\sigma + 1} < 1 \quad \Rightarrow \quad |z_{1,2}| < 1.$$

Combining these two expressions we have the desired unconditional result: $|z_{1,2}| \leq 1$. Since our ansatz was $\hat{v}^n(\omega) = z^n$, then we must have $|\hat{v}^n(\omega)| = |z_{1,2}^n| = |z_{1,2}|^n \leq 1$. In other words, the solution is bounded, so the DuFort-Frankel scheme in Equation 35 is unconditionally stable. In three spatial dimensions, the derivation proceeds in an identical manner. Following the notion on page 77 in [1], we have

$$v_j^n = v(x_j, t_n),\ \text{for } x_j = (j_1h, j_2h, j_3h) \text{ and } t_n = nk.$$

By inspection, the previous characteristic equation will be modified as follows

$$z^2 - \left(\frac{4\sigma}{1+2\sigma}\right)(cos\xi_1 + cos\xi_2 + cos\xi_3)z - \left(\frac{1-2\sigma}{1+2\sigma}\right) = 0$$

Note that the form of the characteristic equation is unchanged, therefore the DuFort-Frankel approximation to $u_t = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$ will also be unconditionally stable. (QED)

References

[1] Bertil Gustafsson, Heinz-Otto Kreiss, and Joseph Oliger. *Time Dependent Problems and Difference Methods*. John Wiley & Sons, 1995.

```
function prob3

% AM 255, Problem Set 4, Problem 3
%    Solves the convection-diffusion equation IVP using
%    a difference approximation. Results are displayed
%    graphically and tabulated for the write-up.
%
% Input:
%    None
%
% Output:
%    Tables/plots required for the write-up.
%
% Douglas Lanman, Brown University, Oct. 2006

% Reset Matlab command window.
clc;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Part I: Specify discrete grid parameters.

% Specify the initial condition and analytic solution at t=1.
IC = @(x) sin(x);
AS = @(x) sin(x-1)/exp(1);

% Set the convection-diffusion parameters.
a      = 1;
nu     = 1;
alpha  = 1;
lambda = 1;

% Define space/time grid interval(s) for evaluation.
N = [10 20 40 80 160 320]; % #gridpoints s.t. N+2 on [0,2*pi]
h = 2*pi./(N+1);           % resulting space steps

% Select time step.
k = (lambda*h)/a;
%k = (alpha*h.^2)./(2*nu);

% Set discrete positions/time-steps for evaluation.
% Note: All time steps will be equal, except the
%       last; it will be adjusted so that the final
%       time will be exactly 1.
for i = 1:length(N)
   x{i} = h(i)*[0:N(i)];
   t{i} = [0:k(i):1];
   if t{i}(end) ~= 1
      t{i}(end+1) = 1;
   end
end
```

```

% Initialize the numerical solution(s).
for i = 1:length(N)
   v{i} = zeros(length(t{i}),N(i)+1);
   v{i}(1,:) = IC(x{i}); % boundary values
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Part II: Solve IVP using difference approximation.

% Update solution sequentially (beginning with I.C.).
% Note: D0.m and DpDm.m implement the difference operators.
for i = 1:length(N)
   for n = 1:(length(t{i})-1)
      v{i}(n+1,:) = v{i}(n,:) + ...
         k(i)*(nu*DpDm(v{i}(n,:),h(i)) - a*D0(v{i}(n,:),h(i)));
   end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Part III: Plot/tabulate modeling results.

% Evaluate the analytic solution.
xe = linspace(0,2*pi,1000);
fe = AS(xe);

% Determine the L2-error and the approximation order.
for i = 1:length(N)
   L2_error(i) = sqrt(sum((abs(AS(x{i})-v{i}(end,:)).^2)*h(i)));
   if i > 1
      order(i) = log2(L2_error(i-1)/L2_error(i));
   end
end

% Tabulate results.
disp(' N     L2-error     order');
disp('---------------------------');
for i = 1:length(N)
   if i > 1
      fprintf('%3d   %.5g   %+2.2f\n',N(i),L2_error(i),order(i));
   else
      fprintf('%3d   %.5g\n',N(i),L2_error(i));
   end
end

% Compare approximation to exact solution.
figure(1); clf;
plot(xe,fe,'r-','LineWidth',3);
hold on;
   plot(x{1},v{1}(end,:),'.','MarkerSize',20);
```

```
hold off;
set(gca,'LineWidth',2,'FontSize',14,'FontWeight','normal');
xlabel('$x_j$','FontName','Times',...
    'Interpreter','Latex','FontSize',16);
%title('Difference Approximation vs. Analytic Solution');
grid on; axis([0 2*pi -2 2]);
legend('Analytic Solution','Difference Approx.');
```

```
function b = D0(a,h)

% DO Central difference operator.
%    DO(A,H) evaluates the central difference of the
%    array A with grid-spacing H, as defined in:
%
%    "Time Dependent Problems and Difference Methods",
%    B. Gustafsson, H.-O. Kreiss, and J. Oliger, 1995.
%
% Douglas Lanman, Brown University, Sept. 2006

% Determine the length of the input array.
N = length(a);

% Shift array indices (modulo the array length).
fj = mod([1:N]-2,N)+1; % shift forward
bj = mod([1:N],N)+1;   % shift backward

% Evaluate the central difference.
b = (a(bj)-a(fj))/(2*h);
```

```
function b = DpDm(a,h)

% DpDm Sequential backward/forward difference operator.
%    DpDm(A,H) evaluates the sequential backward/forward
%    difference of the array A with grid-spacing H, as
%    defined in:
%
%    "Time Dependent Problems and Difference Methods",
%    B. Gustafsson, H.-O. Kreiss, and J. Oliger, 1995.
%
% Douglas Lanman, Brown University, Oct. 2006

% Determine the length of the input array.
N = length(a);

% Shift array indices (modulo the array length).
fj = mod([1:N]-2,N)+1; % shift forward
bj = mod([1:N],N)+1;   % shift backward

% Evaluate the sequential backward/forward difference.
b = (a(bj)-2*a+a(fj))/(h^2);
```