EN 202: Problem Set 1

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Problem 1

Find the general solution to:

$$u_x + u_y = f(x, y),\ u(x, 0) = 0$$
where f is a function defined for $y \geq 0$, $-\infty \leq x \leq \infty$

As discussed in class on 1/27/06, this is a nonhomogeneous constant-coefficient partial differential equation (PDE) of the form $au_x + bu_y = f(x, y)$, where $a = b = 1$. In particular, this question asks us to solve the Cauchy problem – one which is defined on the entire real line. Using the method presented in class, it is a straightforward process to determine the general solution $u(x, y)$. First, notice that the PDE can be expressed as a constraint on the component of the gradient in the direction of the vector $\mathbf{v}$, defined as follows

$$\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, the gradient of u is given by

$$\nabla u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

As a result, we can restate the PDE as $\mathbf{v} \cdot \nabla u = f(x, y)$. This motivates the change of coordinates presented in class

$$\bar{x}(\xi, \eta) \equiv a\xi - b\eta = \xi - \eta \tag{1}$$

$$\bar{y}(\xi, \eta) \equiv b\xi + a\eta = \xi + \eta \tag{2}$$

Applying the chain rule, we find the following relationship

$$\bar{u}(\xi, \eta) = u(\bar{x}, \bar{y})$$

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial \bar{y}}{\partial \xi} = u_x + u_y = f(x, y)$$

It is apparent that the change of basis has been particularly advantageous – converting our PDE to an ODE in ξ (along the characteristic lines of constant η). As a result we have

$$\bar{u}_\xi = \bar{f}, \text{ where } \bar{f}(\xi, \eta) = f(\bar{x}, \bar{y})$$

We can integrate both sides of this expression to obtain the general form of the solution

$$\bar{u}(\xi, \eta) = \phi(\eta) + \int_0^\xi \bar{f}(z, \eta)dz \tag{3}$$

Notice that we have not yet applied the initial condition $u(x,0) = 0$, which defines the curve Γ. In order to apply this condition, we must transform it from (x,y)-coordinates to (ξ,η)-coordinates. Along the curve Γ, $y = 0$ and, substituting into Equation 2, we have $\xi = -\eta$. We can apply this result to Equation 3 to obtain a solution for $\phi(\eta)$.

$$\bar{u}(-\eta,\eta) = 0 = \phi(\eta) + \int_0^{-\eta} \bar{f}(z,\eta)dz \Rightarrow \phi(\eta) = -\int_0^{-\eta} \bar{f}(z,\eta)dz \tag{4}$$

Combining Equations 3 and 4 we obtain the general solution

$$\bar{u}(\xi,\eta) = \int_0^{\xi} \bar{f}(z,\eta)dz - \int_0^{-\eta} \bar{f}(z,\eta)dz$$

As a final step we must determine the transformation to convert from (ξ,η)-coordinates to (x,y)-coordinates. This can be done as follows. First, let

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

now multiply by the left inverse of the matrix to obtain

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In conclusion, we find the general solution is given in the (ξ,η) coordinate system as follows (with the coordinate transform back to the (x,y) system as provided).

$$\boxed{\begin{aligned} \bar{u}(\xi,\eta) &= \int_0^{\xi} \bar{f}(z,\eta)dz - \int_0^{-\eta} \bar{f}(z,\eta)dz \\ \xi &= \frac{x+y}{2}, \eta = \frac{y-x}{2} \end{aligned}} \tag{5}$$

Problem 2

For Problem 1, find the solution for the following functions f. In each case, make a single graph showing f as a function of x for $f(x,0), f(x,1), f(x,2), f(x,3)$, and $f(x,4)$. Make a second graph showing u as a function of x for $u(x,0), u(x,1), u(x,2), u(x,3)$, and $u(x,4)$. Comment on the differences in the nature of the solutions.

(a) $f(x,y) = e^{\frac{1}{2}(x-y)^2}$
(b) $f(x,y) = e^{\frac{1}{2}(x+y)^2}$

Part (a)

To apply Equation 5 we must first transform $f(x,y)$ to the (ξ,η) coordinate system using Equations 1 and 2.

$$\bar{f}(\xi,\eta) = e^{\frac{1}{2}(\bar{x}-\bar{y})^2} = e^{\frac{1}{2}[(\xi-\eta)-(\xi+\eta)]^2} = e^{2\eta^2}$$

Now we can substitute for $\bar{f}(\xi,\eta)$ in Equation 5.

$$\bar{u}(\xi,\eta) = \int_0^{\xi} e^{2\eta^2} dz - \int_0^{-\eta} e^{2\eta^2} dz$$

Note that, in both cases, the integrand is independent of the argument of integration (and can be brought outside the integral).

$$\bar{u}(\xi,\eta) = e^{2\eta^2} \int_0^{\xi} dz - e^{2\eta^2} \int_0^{-\eta} dz = (\xi+\eta)e^{2\eta^2}$$

At this point, we can apply the transformation to the (x, y) coordinate system (given in Problem 1) to obtain the solution for u.

$$u(x,y) = \bar{u}\left(\frac{x+y}{2}, \frac{y-x}{2}\right) = \left(\frac{x+y}{2} + \frac{y-x}{2}\right) \exp\left[2\left(\frac{y-x}{2}\right)^2\right]$$

Simplifying, we obtain

$$\boxed{u(x,y) = ye^{\frac{1}{2}(y-x)^2}, \text{ for } y \geq 0, -\infty \leq x \leq \infty} \tag{6}$$

Plots of $f(x, y)$ and $u(x, y)$ were generated in *Mathematica* and are shown in Figure 1. The *Mathematica* notebook is included at the end of this write-up. Note that, in the figures, the cases $y = \{0, 1, 2, 3, 4\}$ are shown in {red, yellow, green, blue, purple}, respectively.

Part (b)

We can proceed as in Part (a); the nonhomogeneous term $f(x, y)$ is written in (ξ, η) coordinates as

$$\bar{f}(\xi,\eta) = e^{\frac{1}{2}(\bar{x}+\bar{y})^2} = e^{\frac{1}{2}[(\xi-\eta)+(\xi+\eta)]^2} = e^{2\xi^2}$$

Once again, we substitute for $\bar{f}(\xi,\eta)$ in Equation 5.

$$\bar{u}(\xi,\eta) = \int_0^{\xi} e^{2z^2} dz - \int_0^{-\eta} e^{2z^2} dz$$

Consider the change of variables: $\alpha = \sqrt{2}z$ and $\frac{d\alpha}{dz} = \sqrt{2}dz$. Applying this transformation to the previous equation we obtain

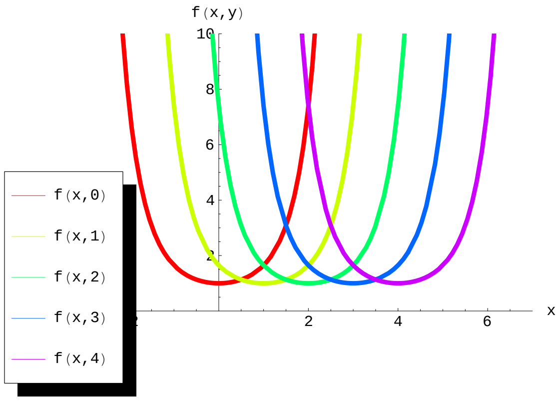
$$\bar{u}(\xi,\eta) = \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}\xi} e^{\alpha^2} d\alpha - \frac{1}{\sqrt{2}} \int_0^{-\sqrt{2}\eta} e^{\alpha^2} d\alpha$$

From [1] we know that the "imaginary error function" $erfi(z)$ is defined as follows

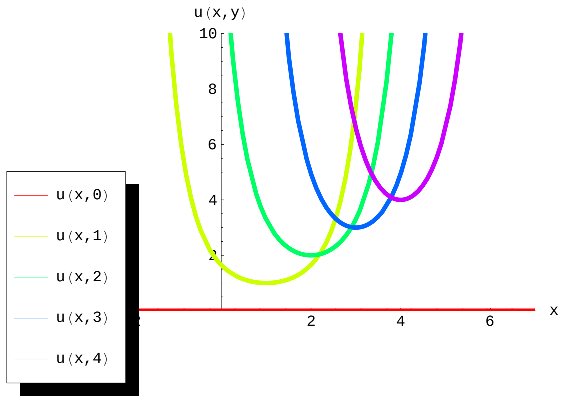
$$\left(\frac{\sqrt{\pi}}{2}\right) erfi(z) = \int_0^{z} e^{\tau^2} d\tau$$

Substituting this result into the previous equation, we find the following expression

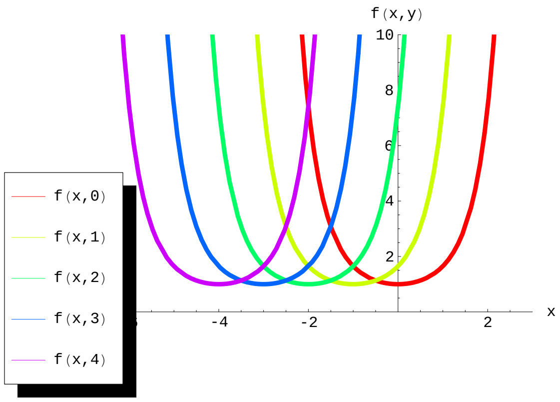
$$\bar{u}(\xi,\eta) = \frac{1}{2}\sqrt{\frac{\pi}{2}} \left[erfi(\sqrt{2}\xi) + erfi(\sqrt{2}\eta)\right]$$



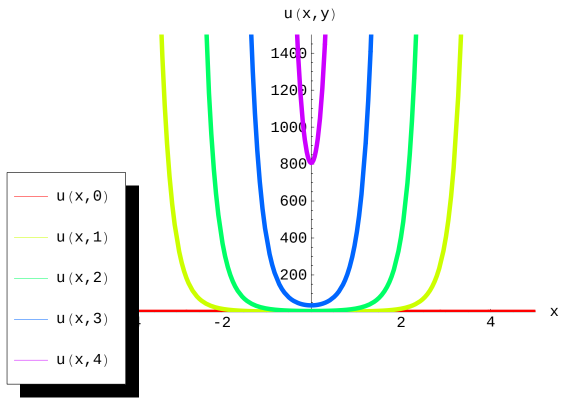
(a) Problem 2(a): $f(x, y)$ for $y = \{0, 1, 2, 3, 4\}$



(b) Problem 2(a): $u(x, y)$ for $y = \{0, 1, 2, 3, 4\}$



(c) Problem 2(b): $f(x, y)$ for $y = \{0, 1, 2, 3, 4\}$



(d) Problem 2(b): $u(x, y)$ for $y = \{0, 1, 2, 3, 4\}$

Figure 1: Comparison of $f(x, y)$ and $u(x, y)$ for several values of y.

Notice that we have used the identity: $erfi(-z) = -erfi(z)$. At this point, we can apply the transformation to the (x, y) coordinate system (given in Problem 1) to obtain the solution for u.

$$u(x, y) = \bar{u}\left(\frac{x+y}{2}, \frac{y-x}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[erfi\left(\frac{x+y}{\sqrt{2}}\right) + erfi\left(\frac{y-x}{\sqrt{2}}\right)\right]$$

In conclusion, we find

$$\boxed{u(x, y) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left[erfi\left(\frac{x+y}{\sqrt{2}}\right) + erfi\left(\frac{y-x}{\sqrt{2}}\right)\right], \text{ for } y \geq 0, -\infty \leq x \leq \infty} \tag{7}$$

Plots of $f(x, y)$ and $u(x, y)$ were generated in *Mathematica* and are shown in Figure 1. The *Mathematica* notebook is included at the end of this write-up. Note that, in the figures, the cases $y = \{0, 1, 2, 3, 4\}$ are shown in {red, yellow, green, blue, purple}, respectively.

Comment on Differences

As can be seen in Figure 1, the form of $f(x, y)$ is similar in Parts (a) and (b). In Part (a), the minimum of $f(x, y)$ is located at $x = y$; that is, $f(x, y)$ is a shifted version of $f(x, 0)$ which has moved y units to the right. Similarly, in Part (b) $f(x, y)$ has a minimum at $x = -y$, so it corresponds to a "time-reversed" version of the function in Part (a) – if y is considered to be the time variable.

The effect of "time-reversal" in this case is dramatic. As can be seen in Figure 1(b), for Part (a) $u(x, 0)$ is the line $y = 0$, however for $y > 0$ $u(x, y)$ represents a function which is moving to the right with a minimum value that is monotonically increasing. In contrast, for Part (b) the minimum of $u(x, y)$ is always located on $x = 0$, but the minimum value increases at a faster rate than in Part (a). This is shown in Figure 1(d). It is important to note that for $y = 0$, $u(x, 0) = 0$ in both Part (a) and Part (b), since this case corresponds to the initial condition provided in Problem 1.

Problem 3

Let k be a positive constant. Find the general solution to the initial value problem:

$$(1 + kx)u_x + u_y = 0, -\infty \leq x \leq \infty, y \geq 0.$$
$$u(x, 0) = u_0(x) \text{ is a given function for } -\infty \leq x \leq \infty$$

As discussed in class on 2/3/06, this is a homogeneous first-order PDE with nonconstant coefficients of the form $au_x + bu_y + cu = f$, where $\{a, b, c, f\}$ are functions of x and y. We want to solve the Cauchy problem, in which the value of $u(x, y)$ is prescribed on some curve Γ in R (Note that, in this case, Γ is the entire real line). As discussed in class on 1/30/06, the solution to this initial value problem will be valid in some region D (possibly empty) of R that contains Γ. To begin, let's define the vector field $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} a(x, y) \\ b(x, y) \end{bmatrix} = \begin{bmatrix} 1 + kx \\ 1 \end{bmatrix}$$

Similarly, the gradient of u is given by

$$\nabla u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

As before, we can restate the PDE as $\mathbf{v} \cdot \nabla u = 0$. Once again, this equation represents a constraint on the component of the gradient of $u(x, y)$ which is parallel to $\mathbf{v}$. To proceed, we want to find a set of curves (i.e. characteristics) $\{\bar{x}(\xi, \eta), \bar{y}(\xi, \eta)\}$, with tracing parameter ξ, such that

$$\{\bar{x}_\xi = a(\bar{x}, \bar{y}) = 1 + k\bar{x}, \bar{y}_\xi = b(\bar{x}, \bar{y}) = 1\} \tag{8}$$

Note that, for any fixed η, the curve traced by ξ is everywhere tangent to the vector field $\mathbf{v}$, by construction. At this point, we want to modify our characteristics such that $\xi = 0$

corresponds to the intersection of the characteristic with the initial value curve Γ. That can be done by choosing

$$\bar{x}(0,\eta) = x_0(\eta) = \eta, \text{ for } -\infty \leq \eta \leq \infty$$

$$\bar{y}(0,\eta) = y_0(\eta) = 0$$

Before we determine the explicit form of the characteristics, let's solve for the general form of u. Applying the chain rule we find

$$U(\xi,\eta) = u(\bar{x},\bar{y})$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial \bar{y}}{\partial \xi} = (1+kx)u_x + u_y = 0$$

Integrating with respect to ξ, we obtain $U(\xi,\eta) = \phi(\eta)$. At this point, we can apply the initial condition on Γ to solve for $\phi(\eta)$. Since $U(0,\eta) = u_0(\eta)$, we find the general solution to the PDE is given by

$$U(\xi,\eta) = u_0(\eta), \text{ for } -\infty \leq \eta \leq \infty \tag{9}$$

To complete our analysis, we must solve Equation 8 to obtain a closed-form expression for the characteristics. Let's begin by integrating $\bar{y}_\xi = 1$ to obtain

$$\bar{y}(\xi,\eta) = \int_0^\xi dz + \phi(\eta) = \xi + \phi(\eta)$$

If we apply the initial condition $\bar{y}(0,\eta) = 0$ we find $\phi(\eta) = 0$. In conclusion, the $\bar{y}$-component of the characteristics is given by

$$\bar{y}(\xi,\eta) = \xi$$

To determine $\bar{x}(\xi,\eta)$ we can solve the ODE $\bar{x}_\xi - k\bar{x} = 1$ given in Equation 8. First, note that the homogeneous solution is given by $\bar{x}(\xi,\eta) = A(\eta)e^{k\xi}$. Using the method of "variation of parameters", we can obtain the general solution as follows. First, substitute $\bar{x}(\xi,\eta) = A(\xi,\eta)e^{k\xi}$ into $\bar{x}_\xi = 1 + k\bar{x}$.

$$A_\xi(\xi,\eta)e^{k\xi} + kA(\xi,\eta)e^{k\xi} = kA(\xi,\eta)e^{k\xi} + 1$$

$$\Rightarrow A_\xi(\xi,\eta) = e^{-k\xi}$$

Integrating with respect to ξ, we find

$$A(\xi,\eta) = \int_0^\xi e^{-kz}dz + \phi(\eta)$$

$$\Rightarrow A(\xi,\eta) = \frac{1}{k}\left(1 - e^{-k\xi}\right) + \phi(\eta)$$

$$\Rightarrow \bar{x}(\xi,\eta) = \left(\phi(\eta) + \frac{1}{k}\right)e^{k\xi} - \frac{1}{k}$$

Substituting the initial condition $\bar{x}(0,\eta) = \eta$, we conclude that $\phi(\eta) = \eta$. In conclusion, the characteristics are given by

$$\bar{x}(\xi,\eta) = \left(\eta + \frac{1}{k}\right)e^{k\xi} - \frac{1}{k} \tag{10}$$

$$\bar{y}(\xi, \eta) = \xi \tag{11}$$

Notice that, at this point, we'd like to determine in what domain D the coordinate transformation can be inverted. As was presented in class on 2/1/06, a necessary condition for the coordinate transformation to be invertible is that the Jacobian determinant is nonzero.

$$\det \begin{pmatrix} \bar{x}_\xi & \bar{x}_\eta \\ \bar{y}_\xi & \bar{y}_\eta \end{pmatrix} \neq 0$$

From Equation 11 we know that $\bar{y}_\xi = 1$ and $\bar{y}_\eta = 0$, so we have

$$\det \begin{pmatrix} \bar{x}_\xi & \bar{x}_\eta \\ \bar{y}_\xi & \bar{y}_\eta \end{pmatrix} = -\bar{x}_\eta = e^{k\xi} \neq 0, \text{ for } \{k, \xi\} \in \mathbb{R}$$

So we find that the coordinate transformation is invertible everywhere in the domain of interest $-\infty \leq x \leq \infty, y \geq 0$ and is given by

$$\bar{\xi}(x, y) = y \tag{12}$$

$$\bar{\eta}(x, y) = \left(x + \frac{1}{k}\right) e^{-ky} - \frac{1}{k} \tag{13}$$

In conclusion, we can substitute for η in Equation 9 to find the general solution for $u(x, y)$.

$$\boxed{u(x, y) = u_0 \left[\left(x + \frac{1}{k}\right) e^{-ky} - \frac{1}{k}\right], \text{ for } -\infty \leq x \leq \infty, y \geq 0} \tag{14}$$

Problem 4

For Problem 3, take $k = 1/\sqrt{2\pi}$ and $u_0(x) = e^{-\frac{1}{2}x^2}$

(a) Find the solution $u(x, y)$ and make a single graph of u versus x which shows $u(x, 0)$, $u(x, 1)$, $u(x, 2)$, $u(x, 3)$, and $u(x, 4)$.

(b) Determine the speed and acceleration of the peak as a function of time and as a function of position.

Part (a)

The specific solution for $u(x, y)$ can be found by substituting for $u_0(x)$ and k in Equation 14.

$$\boxed{u(x, y) = \exp\left(-\frac{1}{2}\left[\left(x + \sqrt{2\pi}\right) e^{-\frac{y}{\sqrt{2\pi}}} - \sqrt{2\pi}\right]^2\right), \text{ for } -\infty \leq x \leq \infty, y \geq 0} \tag{15}$$

Plots of $u(x, y)$ were generated in *Mathematica* and are shown in Figure 2. The *Mathematica* notebook is included at the end of this write-up. Note that, in the figures, the cases $y = \{0, 1, 2, 3, 4\}$ are shown in {red, yellow, green, blue, purple}, respectively.

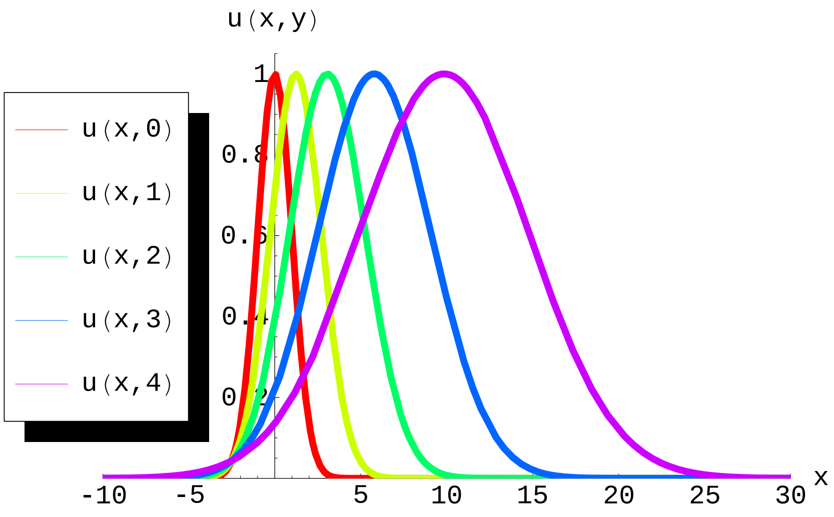


Figure 2: Graph of $u(x,y)$ versus x for y = {0,1,2,3,4} for Problem 4(a).

Part (b)

For this problem we are asked to find the speed and acceleration of the peak as a function of time and position. To begin, we must first determine a closed-form expression for $x_{\text{peak}}(y)$ (the position of the peak as a function of "time" y). Since $u(x,y) = e^{-\frac{1}{2}f(x,y)^2}$, where $f(x,y) = \left(x+\sqrt{2\pi}\right)e^{-\frac{y}{\sqrt{2\pi}}} - \sqrt{2\pi}$, we can differentiate with respect to x to find the peak (where the x-derivative must be zero).

$$\frac{d}{dx}\left(e^{-\frac{1}{2}f(x,y)^2}\right) = \left(e^{-\frac{1}{2}f(x,y)^2}\right) f(x,y)\frac{d}{dx}f(x,y) = 0$$
$$\Rightarrow f(x,y) = \left(x+\sqrt{2\pi}\right)e^{-\frac{y}{\sqrt{2\pi}}} - \sqrt{2\pi} = 0$$

Note that this matches our intuition that the peak of a Gaussian function occurs where the argument to the exponential is equal to zero. Solving for x gives the result

$$x_{\text{peak}}(y) = \sqrt{2\pi}\left(e^{\frac{y}{\sqrt{2\pi}}} - 1\right) \tag{16}$$

The speed of the peak, as a function of time, is given by the first derivative (with respect to "time" y) as

$$\boxed{\dot{x}_{\text{peak}}(y) = \frac{dx_{\text{peak}}(y)}{dy} = e^{\frac{y}{\sqrt{2\pi}}}, \text{ for } y \geq 0} \tag{17}$$

Similarly, the acceleration as a function of time is given by the second derivative

$$\boxed{\ddot{x}_{\text{peak}}(y) = \frac{dx_{\text{peak}}(y)}{dy^2} = \frac{1}{\sqrt{2\pi}}e^{\frac{y}{\sqrt{2\pi}}}, \text{ for } y \geq 0} \tag{18}$$

To determine expressions for the speed and acceleration as a function of peak position, we can solve Equation 16 for y.

$$y = \sqrt{2\pi}\ln\left(\frac{x_{\text{peak}}}{\sqrt{2\pi}} + 1\right), \text{ for } x_{\text{peak}} \geq 0 \text{ and } y \geq 0$$

Substituting this expression into Equations 17 and 18, we find the following results. First, the speed as a function of peak position is given by

$$\boxed{\dot{x}_{\text{peak}}(x_{\text{peak}}) = \frac{x_{\text{peak}}}{\sqrt{2\pi}} + 1, \text{ for } x_{\text{peak}} \geq 0 \text{ and } y \geq 0} \tag{19}$$

Second, the acceleration as a function of peak position is given by

$$\boxed{\ddot{x}_{\text{peak}}(x_{\text{peak}}) = \frac{x_{\text{peak}}}{2\pi} + \frac{1}{\sqrt{2\pi}}, \text{ for } x_{\text{peak}} \geq 0 \text{ and } y \geq 0} \tag{20}$$

Problem 5

Solve the initial value problem

$$yu_x - xu_y = 0$$
$$u(x,0) = u_0(x) \text{ is a given function for } \tfrac{1}{2} \leq x \leq 1$$

Sketch the characteristic curves and show the solution's domain of validity in the (x,y)-plane.

As discussed in class on 2/3/06, this is a homogeneous first-order PDE with nonconstant coefficients of the form $au_x + bu_y + cu = f$, where $\{a,b,c,f\}$ are functions of x and y. We want to solve the Cauchy problem, in which the value of $u(x,y)$ is prescribed on some curve Γ in R. As discussed in class on 1/30/06, the solution to this initial value problem will be valid in some region D (possibly empty) of R that contains Γ. To begin, let's define the vector field $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} a(x,y) \\ b(x,y) \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

Similarly, the gradient of u is given by

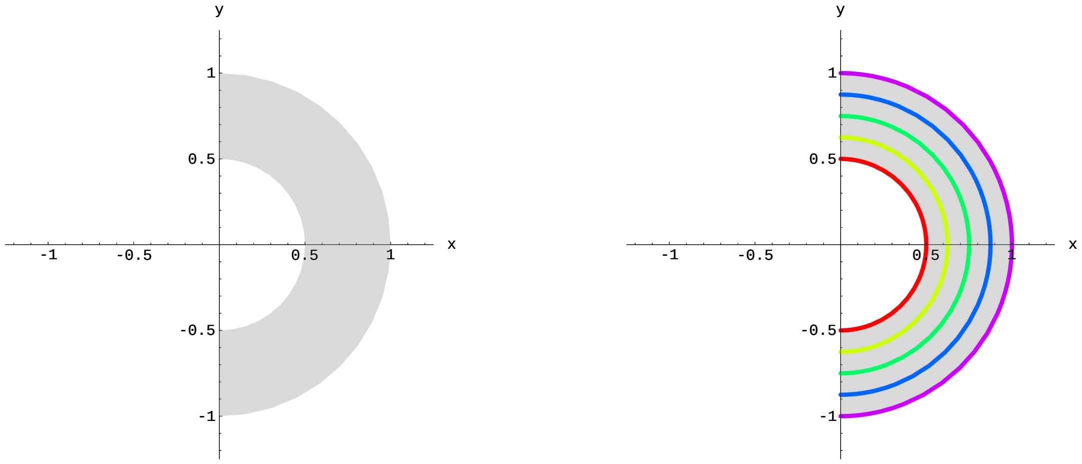
$$\nabla u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

As before, we can restate the PDE as $\mathbf{v} \cdot \nabla u = 0$. Once again, this equation represents a constraint on the component of the gradient of $u(x,y)$ which is parallel to $\mathbf{v}$. To proceed, we want to find a set of curves (i.e. characteristics) $\{\bar{x}(\xi,\eta), \bar{y}(\xi,\eta)\}$, with tracing parameter ξ, such that

$$\{\bar{x}_\xi = a(\bar{x},\bar{y}) = \bar{y}, \bar{y}_\xi = b(\bar{x},\bar{y}) = -\bar{x}\} \tag{21}$$

Note that, for any fixed η, the curve traced by ξ is everywhere tangent to the vector field $\mathbf{v}$, by construction. At this point, we want to modify our characteristics such that $\xi = 0$ corresponds to the intersection of the characteristic with the initial value curve Γ. This can be achieved by choosing

$$\bar{x}(0,\eta) = x_0(\eta) = \eta, \text{ for } \frac{1}{2} \leq \eta \leq 1$$



(a) domain of validity D for $u(x, y)$ (b) characteristic curves for several values of η

Figure 3: Domain of validity and sketch of characteristic curves for Problem 5.

$$\bar{y}(0, \eta) = y_0(\eta) = 0$$

Before we determine the explicit form of the characteristics, let's solve for the general form of u. Applying the chain rule we find

$$U(\xi, \eta) = u(\bar{x}, \bar{y})$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial \bar{y}}{\partial \xi} = y u_x - x u_y = 0$$

Integrating with respect to ξ, we obtain $U(\xi, \eta) = \phi(\eta)$. At this point, we can apply the initial condition on Γ to solve for $\phi(\eta)$. Since $U(0, \eta) = u_0(\eta)$, we find the general solution to the PDE is given by

$$U(\xi, \eta) = u_0(\eta), \text{ for } \frac{1}{2} \leq \eta \leq 1 \tag{22}$$

To complete our analysis, we must solve Equation 21 to obtain a closed-form expression for the characteristics. Let's begin by taking the derivative of $\bar{x}_\xi$ with respect to ξ and substituting for $\bar{x}$ as follows.

$$\bar{x}_{\xi\xi} = \bar{y}_\xi = -\bar{x}$$

The general solution of this ODE is given by

$$\bar{x}(\xi, \eta) = A(\eta)\sin(\xi) + B(\eta)\cos(\xi)$$

Applying the initial condition $\bar{x}(0, \eta) = \eta$, we find $B(\eta) = \eta$. Substituting once again into Equation 21, we find

$$\bar{x}_\xi = \bar{y}(\xi, \eta) = A(\eta)\cos(\xi) - \eta\sin(\xi)$$

Using the initial condition $\bar{y}(0, \eta) = 0$, we find $A(\eta) = 0$, so we conclude that the characteristics are given by

$$\bar{x}(\xi, \eta) = \eta\cos(\xi)$$

$$\bar{y}(\xi, \eta) = -\eta\sin(\xi)$$

in the (ξ, η) coordinate system. To visualize the characteristics, we can eliminate ξ as follows.

$$\bar{x}^2 + \bar{y}^2 = \eta^2(\cos^2(\xi) + \sin^2(\xi)) = \eta^2$$

Substituting $\eta = \sqrt{x^2 + y^2}$, we find the general solution to the PDE is given by

$$\boxed{u(x, y) = u_0(\sqrt{x^2 + y^2}), \text{ for } \frac{1}{2} \leq \sqrt{x^2 + y^2} \leq 1 \text{ and } x \geq 0} \tag{23}$$

Note that the domain of validity D, given by $\frac{1}{2} \leq \sqrt{x^2 + y^2} \leq 1$, defines the half annulus with inner radius $\frac{1}{2}$ and outer radius 1 in the first and fourth quadrants. This region is shaded in Figure 3(a). In addition, the characteristic curves are also show within this domain. Note that, in Figure 3(b), the values of $\eta = \{\frac{1}{2}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1\}$ correspond to the colors {red, yellow, green, blue, and purple}, respectively.

References

[1] Eric W. Weisstein. Erfi. `http://mathworld.wolfram.com/Erfi.html`.