EN 202: Problem Set 3

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Problem 1

Given the initial value problem: $u^2u_x + u_y = 0, u(x,0) = u_0(x) = x$ for $x > 0$, find the solution for $y \geq 0$. Draw the characteristic curves for $x > 0, y \geq 0$ and sketch the solution u as a function of x at various values of the time y. Verify that the solution you find satisfies the initial value problem (i.e., the PDE and the initial conditions).

Note: This problem is very similar to Problem 4 in Problem Set 2; as a result, this solution will follow a similar derivation as [1, 2].

As discussed in class on 2/16/06, this problem focuses on the solution of a quasilinear first-order PDE. Notice that our initial curve Γ is the line $y = 0$. As a result, we can parameterize it as

$$\bar{x}(0,\eta) = x_0(\eta) = \eta$$

$$\bar{y}(0,\eta) = y_0(\eta) = 0$$

for $\eta > 0$. In addition, we define the vector field $\mathbf{v}(\xi,\eta)$ as

$$\mathbf{v}(\xi,\eta) = \begin{bmatrix} \bar{x}_\xi(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) \end{bmatrix} = \begin{bmatrix} U^2(\xi,\eta) \\ 1 \end{bmatrix}$$

As usual, the PDE becomes an ODE along the characteristics such that $\mathbf{v} \cdot \nabla u = 0$. Applying the chain rule, we find $U_\xi(\xi,\eta) = 0$. We can integrate this ODE with respect to ξ to determine $U(\xi,\eta) = \phi(\eta)$. Notice that, along Γ, $U(0,\eta) = u_0(\eta) = \eta$. Applying this initial condition, we have

$$U(\xi,\eta) = \eta, \text{ for } \eta > 0 \tag{1}$$

Substituting this result into our expression for $\mathbf{v}(\xi,\eta)$ we find

$$\mathbf{v}(\xi,\eta) = \begin{bmatrix} \bar{x}_\xi(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) \end{bmatrix} = \begin{bmatrix} \eta^2 \\ 1 \end{bmatrix} \tag{2}$$

At this point, we require an explicit form of the characteristics. First, consider the solution to the ODE $\bar{y}_\xi(\xi,\eta) = 1$. Integrating with respect to ξ, we find $\bar{y} = \xi + \phi(\eta)$. Applying the initial value on Γ, where $y_0(\eta) = 0$ and $\xi = 0$, we find $\phi(\eta) = 0$. Next, consider the solution to the ODE $\bar{x}_\xi(\xi,\eta) = \eta^2$. Integrating with respect to ξ, we find $\bar{x} = \xi\eta^2 + \psi(\eta)$. Using the initial value on Γ, where $x_0(\eta) = \eta$ and $\xi = 0$, we find $\psi(\eta) = \eta$. Combining these results with Equation 1, we obtain the following solution to the PDE given by a parametric surface in the (x, y, u)-space, with tracing parameters (ξ,η).

$$\begin{aligned} \bar{x}(\xi,\eta) &= \xi\eta^2 + \eta \\ \bar{y}(\xi,\eta) &= \xi \\ U(\xi,\eta) &= \eta, \text{ for } n > 0 \end{aligned} \tag{3}$$

We can substitute y for ξ to find

$$x = y\eta^2 + \eta \Rightarrow y\eta^2 + \eta - x = 0$$

Solving for η using the quadratic formula, we find

$$\eta = \frac{-1 \pm \sqrt{1+4xy}}{2y}$$

Substituting into Equation 3, we have

$$u(x,y) = \frac{-1 \pm \sqrt{1+4xy}}{2y} \text{ for } x > 0, y \geq 0$$

We are not finished; we must still determine whether the positive or negative square root is appropriate for the region $x > 0, y \geq 0$. Consider the limit at $y \to 0$; if we allow the negative square root, then there will be division by zero. As a result, only the positive square root can be allowed:

$$\lim_{y \to 0} \frac{-1 + \sqrt{1+4xy}}{2y} = \lim_{y \to 0} \frac{x}{\sqrt{1+4xy}} = x$$

In conclusion, the solution to the initial value problem is given by

$$\boxed{u(x,y) = \frac{-1 + \sqrt{1+4xy}}{2y} \text{ for } x > 0, y \geq 0} \tag{4}$$

Note that we have already verified that this solution satisfies the initial conditions (i.e., that $\lim_{y\to 0} u(x,y) = u_0(x) = x$). At this point, we must verify that Equation 4 satisfies the PDE. First, let's evaluate u^2.

$$u^2(x,y) = \left(\frac{-1 + \sqrt{1+4xy}}{2y}\right)^2 = \frac{1 - \sqrt{1+4xy} + 2xy}{2y^2}$$

Similarly, u_x is given by

$$u_x(x,y) = \frac{1}{\sqrt{1+4xy}}$$

Finally, u_y is given by

$$u_y(x,y) = \frac{x}{y\sqrt{1+4xy}} + \frac{1 - \sqrt{1+4xy}}{2y^2} = \frac{-1 + \sqrt{1+4xy} - 2xy}{2y^2\sqrt{1+4xy}}$$

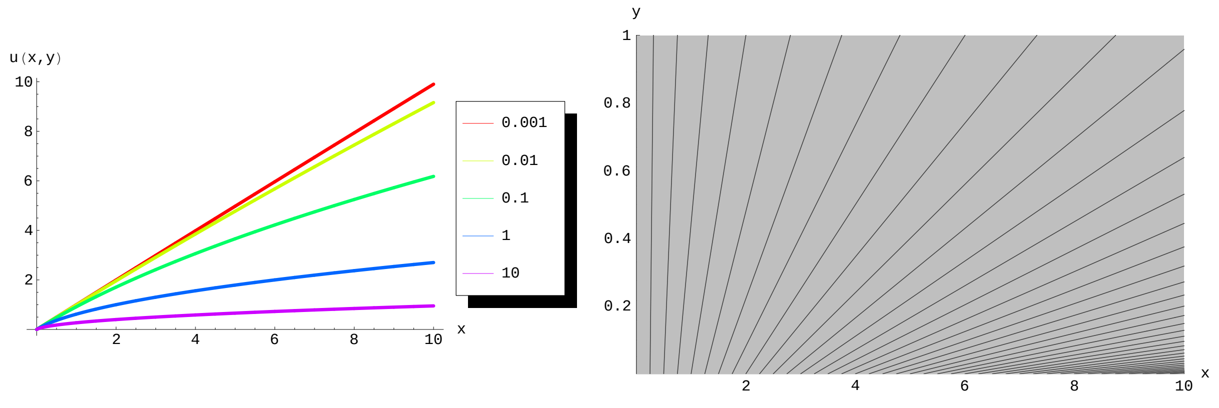
Substituting these results into the PDE $u^2 u_x + u_y = 0$, we see that the solution for $u(x,y)$ satisfies the PDE.

$$u^2 u_x = -u_y = \frac{1 - \sqrt{1+4xy} + 2xy}{2y^2\sqrt{1+4xy}}$$

Returning to Equation 3, we find that the characteristic curves are given by

$$\boxed{y = \left(\frac{1}{\eta^2}\right) x - \frac{1}{\eta}} \tag{5}$$

Plots of this specific solution were generated in *Mathematica* and are shown in Figure 1(a). The *Mathematica* notebook is included at the end of this write-up. Note that, in the figures, the cases $y = \{0.001, 0.01, 0.1, 1, 10\}$ are shown in {red, yellow, green, blue, purple}, respectively. Similarly, the characteristics given by Equation 5 are shown in Figure 1(b) (the domain of validity $x > 0, y \geq 0$ is shaded in gray).



(a) illustration of $u(x, y)$ for various values of y (b) characteristic curves for several values of η

Figure 1: Illustration of $u(x, y)$ and characteristics found in Problem 1.

Problem 2

Recall Problem 4 from Problem Set 2:

$$uu_x + u_y = 0, u(x, 0) = \sin(x)$$

Plot the characteristics for this IVP for the region $-10 < x < 10$ and $0 < y < 3$. Prove that the earliest time of intersection of the characteristics is at $y = 1$. Find the corresponding positions of the wave break.

From [1, 2] we know that the characteristics, indexed by η, are given by

$$\bar{x}(\xi, \eta) = \xi \sin(\eta) + \eta$$

$$\bar{y}(\xi, \eta) = \xi$$

As a result, we can plot the characteristics for this IVP for the region $-10 < x < 10$ and $0 < y < 3$ – as has been done in Figure 2. Note that, graphically, we observe that the earliest time that the characteristics intersect occurs around $y = 1$, confirming our expectation that this is the time at which wave-breaking occurs. We can prove this explicitly by following the method presented in class on 2/15/06. First, note that the equations for the characteristics can be combined as follows.

$$x = y \sin(\eta) + \eta$$

Consider two characteristics (indexed by η_1 and η_2) which intersect at the point (x, y).

$$x = y \sin(\eta_1) + \eta_1$$

$$x = y \sin(\eta_2) + \eta_2$$

Subtracting the second equation from the first, we find

$$y \sin(\eta_1) + \eta_1 - y \sin(\eta_2) - \eta_2 = 0$$

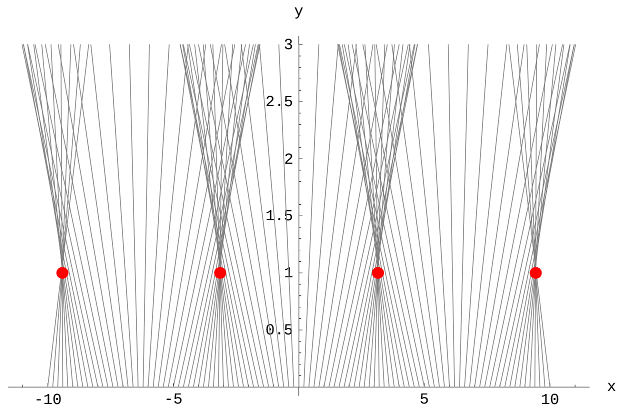


Figure 2: Wave-breaking points and characteristics found in Problem 2.

$$\Rightarrow (\sin(\eta_2) - \sin(\eta_1))y = \eta_1 - \eta_2 \Rightarrow y = \frac{\eta_1 - \eta_2}{\sin(\eta_2) - \sin(\eta_1)}$$

We can determine the earliest time of intersection y by taking the limit as $\eta_2 \to \eta_1$. Note that this limit corresponds to the time at which an infinitesimal change in η results in an intersection of the characteristics at time y.

$$y = \lim_{\eta_2 \to \eta_1} \frac{\eta_1 - \eta_2}{\sin(\eta_2) - \sin(\eta_1)} = \frac{-1}{\cos(\eta_1)}$$

Note that L'Hospital's Rule has been applied in this derivation. It is important to note that the result $y = -1/\cos(\eta)$ was found in Problem 4 from Problem Set 2 using different methods in [1, 2]. To determine the earliest time y of wave-breaking, we proceed as in [1]. Notice that the following constraints on (x, y) must hold.

$$y\cos(\eta) = -1 \tag{6}$$

$$y\sin(\eta) = x - \eta \tag{7}$$

Combining these two equations we find

$$(y\cos(\eta))^2 + (y\sin(\eta))^2 = y^2 = 1 + (x-\eta)^2$$

$$\Rightarrow \eta = x \pm \sqrt{y^2 - 1} \tag{8}$$

Notice that Equation 8 will, in general, have no real-valued solutions for $0 \le y < 1$, one solution for $y = 1$, and two solutions for $y > 1$. As a result, the earliest time of wave-breaking (i.e., intersection of the characteristics) corresponds to the "transition time" y = 1. (QED)

To complete our analysis, we can substitute $y = 1$ into Equation 8 to find $\eta = x$ at the wave-breaking points. Substituting $\eta = x$ and $y = 1$ into Equation 6, we find that $\cos(x) = -1$ at the wave-breaking points; as a result, the wave-breaking points are given by

$$\boxed{\begin{array}{l} \text{wave-breaking time: } y = 1 \\[1em] \text{wave-breaking points: } \{x = \pi(2k+1), k \in \mathbb{Z}, y = 1\} \end{array}} \tag{9}$$

In conclusion, we find that the solution found previously in [1, 2] can also be obtained by considering the earliest time that a pair of characteristics intersect.

Problem 3

For the IVP $(1+ku)u_x + u_y = 0, u(x_0(\eta), y_0(\eta)) = u_0(\eta)$ for $\eta_1 \leq \eta \leq \eta_2$, show that the initial curve Γ: $\{x = x_0(\eta), y = y_0(\eta), \eta_1 \leq \eta \leq \eta_2\}$ is nowhere tangent to a characteristic if and only if $\dot{y}_0(\eta)(1+ku_0(\eta)) \neq \dot{x}_0(\eta)$ for $\eta_1 \leq \eta \leq \eta_2$. Here the "dot" stands for the derivative with respect to η.

Let us begin by finding the normal vector $\mathbf{w}(\eta)$ for the initial curve Γ: $\{x = x_0(\eta), y = y_0(\eta), \eta_1 \leq \eta \leq \eta_2\}$ using the equation provided in class on 2/1/06.

$$\mathbf{w}(\eta) = \begin{bmatrix} \dot{y}_0(\eta) \\ -\dot{x}_0(\eta) \end{bmatrix}$$

Recall from class on 2/1/06 that a necessary condition for the (ξ,η)-coordinates to be invertible (i.e., transformed to (x,y)-coordinates) is that the Jacobian determinate $\Delta(0,\eta) \neq 0$ on Γ. Further recall that, on Γ, the Jacobian determinate will be zero anywhere the initial curve Γ is tangent to a characteristic. If we express the vector field in the typical manner as

$$\mathbf{v}(\xi,\eta) = \begin{bmatrix} \bar{x}_\xi(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) \end{bmatrix} = \begin{bmatrix} 1+kU(\xi,\eta) \\ 1 \end{bmatrix} \Rightarrow \mathbf{v}(0,\eta) = \begin{bmatrix} 1+ku_0(\eta) \\ 1 \end{bmatrix}$$

then we can express the condition that Γ is tangent to a characteristic as

$$\mathbf{v}(0,\eta) \cdot \mathbf{w}(\eta) = 0 \Rightarrow \Delta(0,\eta) = 0 \text{ on } \Gamma$$

Substituting for $\mathbf{v}(0,\eta)$ and $\mathbf{w}(\eta)$ we find

$$\begin{bmatrix} 1+ku_0(\eta) & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{y}_0(\eta) \\ -\dot{x}_0(\eta) \end{bmatrix} = \dot{y}_0(\eta)(1+ku_0(\eta)) - \dot{x}_0(\eta) = 0$$

when Γ is tangent to a characteristic – precisely the condition we wish to prove. In conclusion, the initial curve Γ will be nowhere tangent to a characteristic if and only if

$$\boxed{\dot{y}_0(\eta)(1+ku_0(\eta)) \neq \dot{x}_0(\eta), \text{for } \eta_1 \leq \eta \leq \eta_2} \tag{10}$$

Note that we could have arrived at this result by evaluating the Jacobian determinant $\Delta(\xi,\eta)$ directly. Applying the constraint that $\Delta(\xi,\eta) \neq 0$ is required for the coordinate transformation to be invertible, we have

$$\Delta(\xi,\eta) = \det \begin{pmatrix} \bar{x}_\xi(\xi,\eta) & \bar{x}_\eta(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) & \bar{y}_\eta(\xi,\eta) \end{pmatrix} = \begin{pmatrix} 1+kU(\xi,\eta) & \bar{x}_\eta(\xi,\eta) \\ 1 & \bar{y}_\eta(\xi,\eta) \end{pmatrix} \neq 0$$

Evaluating this expression on the initial curve Γ, where $\xi = 0$, we have

$$\Delta(0,\eta) = \det \begin{pmatrix} 1+kU(0,\eta) & \bar{x}_\eta(0,\eta) \\ 1 & \bar{y}_\eta(0,\eta) \end{pmatrix} = \begin{pmatrix} 1+ku_0(\eta) & \dot{x}_0(\eta) \\ 1 & \dot{y}_0(\eta) \end{pmatrix} \neq 0$$

Once again, we find $\dot{y}_0(\eta)(1+ku_0(\eta)) \neq \dot{x}_0(\eta)$ for $\eta_1 \leq \eta \leq \eta_2$ must hold for the initial curve to be nowhere tangent to a characteristic. (QED)

Problem 4

In Problem 3, take Γ: $\{x = x_0(\eta) = \eta, y = y_0(\eta) = \eta, -\infty < \eta < \infty\}$ and $u_0(\eta) = \eta$. Are the characteristics tangent to Γ at any point? If so, is the initial condition compatible with the PDE at the point(s) of tangency? Does a solution exist to the IVP?

To determine if the initial curve Γ, given by the line $y = x$, is tangent to a characteristic at any point, we can evaluate Equation 10 with $x_0(\eta) = \eta$, $y_0(\eta) = \eta$, and $u_0(\eta) = \eta$.

$$\dot{y}_0(\eta)(1 + ku_0(\eta)) \neq \dot{x}_0(\eta), \text{for } -\infty < \eta < \infty$$

$$\Rightarrow 1 + k\eta \neq 1 \Rightarrow k\eta \neq 0$$

Note that we have used the fact that $\dot{x}_0(\eta) = 1$ and $\dot{y}_0(\eta) = 1$. For $k \neq 0$, we find that Γ will be tangent to a characteristic when $\eta = 0$. For $k = 0$, Γ will be tangent everywhere (i.e., for $-\infty < \eta < \infty$).

To proceed, we'll first find closed-form expressions for the characteristics. To begin, let's define the vector field $\mathbf{v}(\xi, \eta)$ as

$$\mathbf{v} = \begin{bmatrix} \bar{x}_\xi(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) \end{bmatrix} = \begin{bmatrix} 1 + kU(\xi,\eta) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + k\eta \\ 1 \end{bmatrix}$$

Consider the solution to the ODE $\bar{y}_\xi(\xi, \eta) = 1$. Integrating with respect to ξ, we find $\bar{y} = \xi + \phi(\eta)$. Applying the initial value on Γ, where $y_0(\eta) = \eta$ and $\xi = 0$, we find $\phi(\eta) = \eta$. Similarly, consider the solution to the ODE $\bar{x}_\xi(\xi, \eta) = 1 + k\eta$. Integrating with respect to ξ, we find $\bar{x} = (1 + k\eta)\xi + \psi(\eta)$. Using the initial value on Γ, where $x_0(\eta) = \eta$ and $\xi = 0$, we find $\psi(\eta) = \eta$. Combining these results, we obtain the following parametric solution for the characteristics.

$$\bar{x}(\xi, \eta) = (1 + k\eta)\xi + \eta \tag{11}$$

$$\bar{y}(\xi, \eta) = \xi + \eta \tag{12}$$

Substituting $\eta = 0$ into Equations 11 and 12, we find the characteristic indexed by $\eta = 0$ is the line $y = x$ (i.e., $\bar{x}(\xi, 0) = \bar{y}(\xi, 0) = \xi$ for $-\infty < \xi < \infty$). Note that, regardless of the value of k, there will always be a characteristic line $y = x$ which is identical to the initial value curve Γ! As a result, we conclude that **the initial value curve $\boldsymbol{\Gamma}$, given by $\mathbf{y = x}$, is tangent to a characteristic ($\eta = \mathbf{0}$) at every point on $\boldsymbol{\Gamma}$**. In other words, Γ is a characteristic.

Recall from class on 1/25/06, when Γ is coincident with a characteristic there are only two outcomes: (1) either there is no solution to the IVP or (2) there are infinitely many solutions. As was shown in class, if $u_0(\eta)$ is equal to a constant u_* on Γ, then there are infinitely many solutions – given by the set $\{U(\xi, \eta)\}$ such that $U(0, \eta) = u_0(\eta) = u_*$ for $u_* \in \mathbb{R}$. As a result, **the initial condition $\mathbf{u_0}(\eta) = \eta$ does not allow any solutions to this IVP**.

We can also demonstrate that there are no solutions to this initial value problem by showing that the initial condition is incompatible with the PDE at the points of tangency. Previously, we found that $\eta = 0$ was a point of tangency. Along the curve Γ, this point is given by the origin $(x, y) = (0, 0)$. Substituting $\eta = 0$ into the PDE we find

$$(1 + k\eta)u_x(x, y) + u_y(x, y) = 0$$

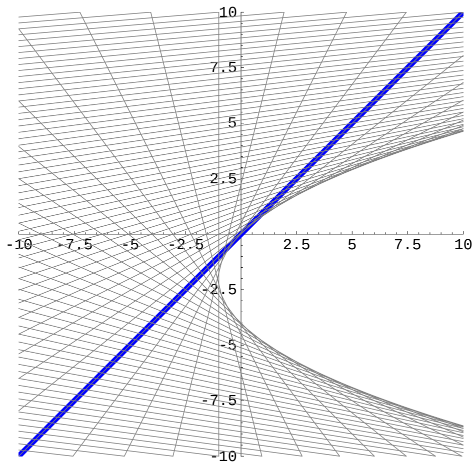


Figure 3: Plot of characteristics (gray) and Γ (blue) for $k = 1$ in Problem 4.

$$\Rightarrow u_x(0,0) + u_y(0,0) = 0$$

Since Γ is the line $y = x$, we have

$$u_0(x,x) = x \Rightarrow u_x(0,0) = 1 \text{ and } u_y(0,0) = 0$$

Note that the initial condition does not satisfy the PDE at the point of tangency $(x, y) = (0, 0)$ (i.e., $u_x(0,0) + u_y(0,0) \neq 0$). As a result, there is no solution to the IVP.

References

[1] Prof. J. A. Blume. Solution Set 2. http://www.engin.brown.edu/courses/en202/homework/hw2/hw2s.pdf.

[2] Douglas R. Lanman. Problem Set 2. http://mesh.brown.edu/dlanman/courses/en202/HW2.pdf.