# EN 202: Problem Set 4

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## Problem 1

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For each of the following functions $u_0(x)$, construct solutions to the initial value problem

$$(1 + ku)u_x + u_y = 0, y \geq 0$$
$$u(x, 0) = u_0(x) \text{ is a given function for } -\infty \leq x \leq \infty$$

Draw the characteristics in the $(x, y)$-plane. If you find the wave-breaking phenomenon arising, determine the time and place of the initial break appearance and find a solution containing a shockwave.

For this first problem, consider $u_0(x)$, for $k \geq 0$ and $\varepsilon > 0$, given by

$$u_0(x) = \begin{cases} 1, & x \geq \varepsilon \\ \frac{x}{\varepsilon}, & 0 < x < \varepsilon \\ 0, & x \leq 0 \end{cases}$$

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As discussed in class on 2/17/06, this problem focuses on the solution of a quasilinear first-order PDE. Note that our initial curve $\Gamma$ is the line $y = 0$. As a result, we can parameterize it as

$$\bar{x}(0, \eta) = x_0(\eta) = \eta$$

$$\bar{y}(0, \eta) = y_0(\eta) = 0$$

In addition, we define the vector field $\mathbf{v}(\xi, \eta)$ as

$$\mathbf{v}(\xi, \eta) = \begin{bmatrix} \bar{x}_\xi(\xi, \eta) \\ \bar{y}_\xi(\xi, \eta) \end{bmatrix} = \begin{bmatrix} 1 + kU(\xi, \eta) \\ 1 \end{bmatrix}$$

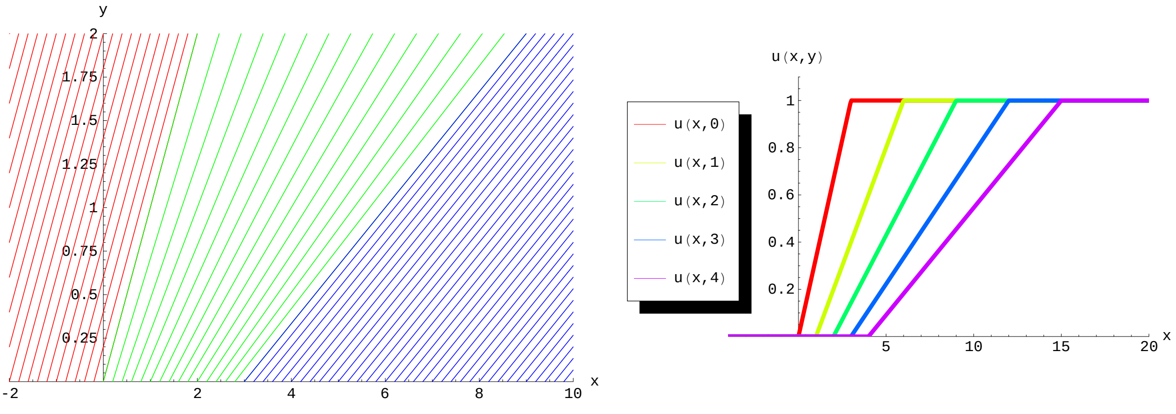
As usual, the PDE becomes an ODE along the characteristics such that $\mathbf{v} \cdot \nabla u = 0$. Applying the chain rule, we find $U_\xi(\xi, \eta) = 0$. We can integrate this ODE with respect to $\xi$ to determine $U(\xi, \eta) = \phi(\eta)$. Notice that, along $\Gamma$, $U(0, \eta) = u_0(\eta)$. Applying this initial condition, we have

$$U(\xi, \eta) = u_0(\eta), \text{ for } -\infty \leq \eta \leq \infty \tag{1}$$

Substituting this result into our expression for $\mathbf{v}(\xi, \eta)$ we find

$$\mathbf{v}(\xi, \eta) = \begin{bmatrix} \bar{x}_\xi(\xi, \eta) \\ \bar{y}_\xi(\xi, \eta) \end{bmatrix} = \begin{bmatrix} 1 + ku_0(\eta) \\ 1 \end{bmatrix} \tag{2}$$

At this point, we require an explicit form of the characteristics. First, consider the solution to the ODE $\bar{y}_\xi(\xi, \eta) = 1$. Integrating with respect to $\xi$, we find $\bar{y} = \xi + \phi(\eta)$. Applying the initial value on $\Gamma$, where $y_0(\eta) = 0$ and $\xi = 0$, we find $\phi(\eta) = 0$. Next, consider the solution to the ODE $\bar{x}_\xi(\xi, \eta) = 1 + ku_0(\eta)$. Integrating with respect to $\xi$, we find $\bar{x} = [1 + ku_0(\eta)]\xi + \psi(\eta)$.




(a) plot of characteristics in each region (b) plot of solution $u(x, y)$ for several times $y$

Figure 1: Characteristics and solution found in Problem 1 (for $k = 2$ and $\varepsilon = 3$).

Using the initial value on $\Gamma$, where $x_0(\eta) = \eta$ and $\xi = 0$, we find $\psi(\eta) = \eta$. Combining these results with Equation 1, we obtain the following solution to the PDE given by a parametric surface in the $(x, y, u)$-space, with tracing parameters $(\xi, \eta)$.

$$\begin{aligned} \bar{x}(\xi, \eta) &= [1 + ku_0(\eta)]\xi + \eta \\ \bar{y}(\xi, \eta) &= \xi \\ U(\xi, \eta) &= u_0(\eta), \text{ for } -\infty \leq \eta \leq \infty \end{aligned} \tag{3}$$

We can substitute $y$ for $\xi$ to find a parametrization of the characteristics in $\eta$

$$x = [1 + ku_0(\eta)]y + \eta, \text{ for } -\infty \leq \eta \leq \infty \tag{4}$$

At this point, we must evaluate the characteristics (and solution) within each region given by $u_0(\eta)$.

**Region I:** $\eta \leq 0$, $u_0(\eta) = 0$

Substituting $u_0(\eta) = 0$ into Equation 4, we find that the characteristics are given by

$$x = y + \eta \Rightarrow \eta = x - y$$

Similarly, the solution within Region I must be

$$u_{\mathrm{I}}(x, y) = 0, \text{ for } x \leq y$$

Note that the characteristics for Region I are sketched in red in Figure 1(a).

**Region II:** $0 < \eta < \varepsilon$, $u_0(\eta) = \frac{\eta}{\varepsilon}$

Substituting $u_0(\eta) = \frac{\eta}{\varepsilon}$ into Equation 4, we find that the characteristics are given by

$$x = \left(1 + \frac{k\eta}{\varepsilon}\right) y + \eta \Rightarrow \eta = \frac{\varepsilon(x - y)}{ky + \varepsilon}$$

Similarly, the solution within Region II must be

$$u_{\mathrm{II}}(x,y) = \frac{x-y}{ky+\varepsilon}, \text{ for } 0 < \frac{\varepsilon(x-y)}{ky+\varepsilon} < \varepsilon$$

Note that the characteristics for Region III are sketched in green in Figure 1(a).

**Region III:** $\eta \geq \varepsilon$, $u_0(\eta) = 1$

Substituting $u_0(\eta) = 1$ into Equation 4, we find that the characteristics are given by

$$x = (1+k)y + \eta \Rightarrow \eta = x - (1+k)y$$

Similarly, the solution within Region III must be

$$u_{\mathrm{III}}(x,y) = 1, \text{ for } x - (1+k)y \geq \varepsilon$$

Note that the characteristics for Region III are sketched in blue in Figure 1(a).

In conclusion, we find that there is no intersection of the characteristics (i.e., the various regions are disjoint). As a result, there is no wave-breaking and the solution is given by

$$\boxed{u(x,y) = \begin{cases} 1, & x-(1+k)y \geq \varepsilon \\ \frac{x-y}{ky+\varepsilon}, & 0 < \frac{x-y}{ky+\varepsilon} < 1 \\ 0, & x \leq y \end{cases}, \text{ for } -\infty \leq x \leq \infty, y \geq 0} \quad (5)$$

# Problem 2

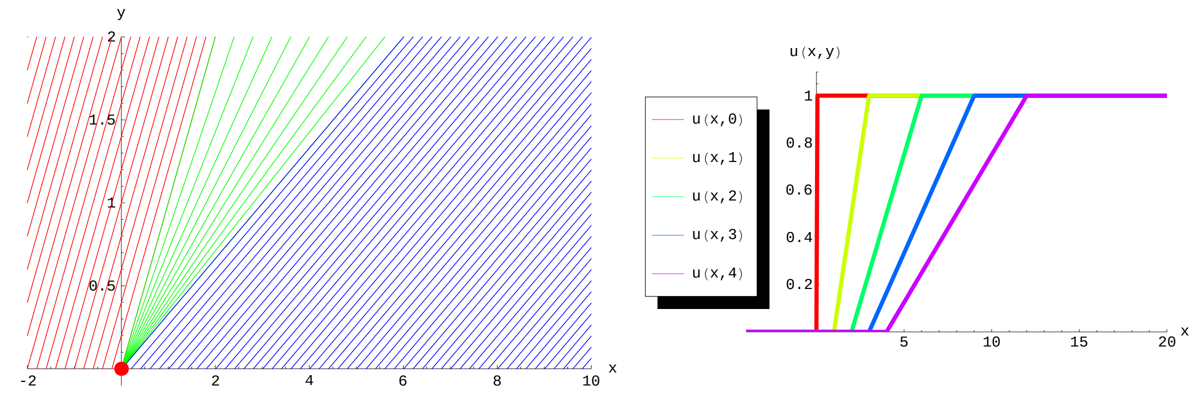
Evaluate the the limit as $\varepsilon \to 0$ for the solution you found in Problem 1. This situtation corresponds to an initial condition $u_0(x)$ equal to the Heaviside step function [3].

From inspecting the plot of $u_0(x)$ in the problem statement, it is apparent that $u(x,0) = u_0(x)$ will initially have a vertical tangent at $x = 0$ (when $y = 0$). Unlike Problem 3, however, we find that this vertical tangent does not give rise to the wave-breaking phenomenon. From class on 2/22/06 we know that, for $y > 0$, the "top" of the wave will propagate with velocity $1+k$, whereas the "bottom" of the wave will propagate with unit velocity 1. As a result, the top of the wave will immediately overtake the bottom for $y > 0$ and the solution for $u(x,y)$ will be single-valued for $y > 0$. This result is sketched in Figure 2(b).

We can derive a closed-form expression for the solution $u(x,y)$ by considering the limit as $\varepsilon \to 0$ in Equation 5. If we exclude the vertical tangent at $(x,y) = (0,0)$, we have

$$\boxed{u(x,y) = \begin{cases} 1, & x-(1+k)y \geq 0 \\ \frac{x-y}{ky}, & 0 < \frac{x-y}{ky} < 1 \\ 0, & x \leq y \end{cases}, \text{ for } -\infty \leq x \leq \infty, y > 0} \quad (6)$$

So, in conclusion, we find that the solution is very similar to before, except that there is an initial discontinuity at $(x,y) = (0,0)$. This singular point is indicated by a red circle in Figure 2(a).




(a) plot of characteristics in each region (b) plot of solution $u(x, y)$ for several times $y$

Figure 2: Characteristics and solution found in Problem 2 (for $k = 2$).

To complete our analysis, we need to obtain expressions for characteristics in each region. First, notice that in Figure 1(a) that the division between Region II and Region III along $y = 0$ is at $x = \varepsilon$. As a result, Region I and Region III will intersect at the origin in the limit as $\varepsilon \to 0$, as shown in Figure 2(a). Similarly, Region II, indicated by the green wedge in the figure, will consist of a set of characteristics which intersect at the origin. To define the characteristics in Region II, we can apply the two constraints: (1) characteristics in Region II are straight lines and (2) characteristics in Region II must intersect the origin. As a result, at any point $(x_0, y_0) \neq (0, 0)$ in Region II, the characteristic is given by the line $y = (\frac{y_0}{x_0})x$, for $y > 0$. The equation for the characteristics in Region I and Region III can be obtained by taking the limit as $\varepsilon \to 0$ for the solutions in Problem 1.

$$\text{Region I Characteristics: } \eta = x - y, \text{ for } x \leq y$$

$$\text{Region III Characteristics: } \eta = x - (1 + k)y, \text{ for } x \geq (1 + k)y$$

The characteristics are sketched in Figure 2(a). Once again, Regions I, II, and III are shown as red, green, and blue lines, respectively.

# Problem 3

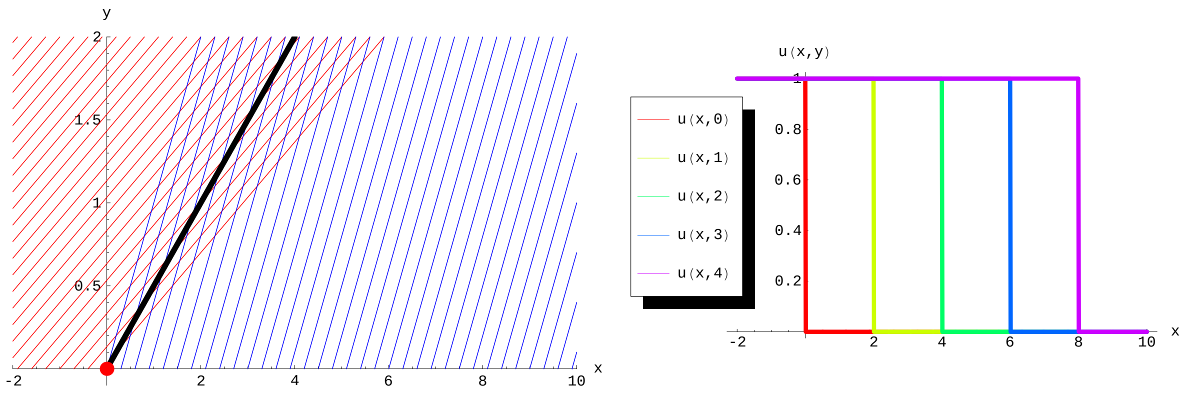
Consider $u_0(x)$, for $k \geq 0$, given by

$$u_0(x) = \begin{cases} 0, & x \geq 0 \\ 1, & x < 0 \end{cases}$$

To begin our analysis, note that there is a vertical tangent for $u(x, 0) = u_0(x)$ located at $x = 0$. As a result, we know that the wave-breaking point is given by $(x, y) = (0, 0)$.

$$\boxed{\text{wave-breaking point: } x = 0, y = 0} \tag{7}$$

Since there is a wave-breaking point, we seek a weak solution of the PDE containing a shock. That is, we want a solution that is single-valued throughout $-\infty \leq x \leq \infty$ *except* at the




(a) characteristics, breakpoint, and shock (b) plot of weak solution $u(x, y)$ for several times $y$

Figure 3: Characteristics and weak solution found in Problem 3 (for $k = 2$).

time-varying position of the shock $x = s(y)$. As was done in Problem 1, we proceed by examining the set of characteristics and solutions within each region defined by the initial condition $u_0(x)$.

**Region I:** $\eta < 0$, $u_0(\eta) = 1$

Substituting $u_0(\eta) = 1$ into Equation 4, we find that the characteristics are given by

$$x = (1 + k)y + \eta \Rightarrow \eta = x - (1 + k)y$$

Similarly, the solution within Region I must be

$$u_{\mathrm{I}}(x, y) = 1, \text{ for } x < (1 + k)y$$

Note that the characteristics for Region I are sketched in red in Figure 3(a).

**Region II:** $\eta \geq 0$, $u_0(\eta) = 0$

Substituting $u_0(\eta) = 0$ into Equation 4, we find that the characteristics are given by

$$x = y + \eta \Rightarrow \eta = x - y$$

Similarly, the solution within Region II must be

$$u_{\mathrm{II}}(x, y) = 0, \text{ for } x \geq y$$

Note that the characteristics for Region II are sketched in blue in Figure 3(a).

First, we notice that the characteristics intersect at the wave-breaking point – consistent with its interpretation as the point where the solution becomes multiple-valued. Also, we immediately notice that there is a wedge, given by $y \leq x < (1 + k)y$ and $y \geq 0$, where the characteristics intersect. In this region the solutions $u_{\mathrm{I}}(x, y)$ and $u_{\mathrm{II}}(x, y)$ both hold. As a result, we must introduce a shock $\mathcal{S}$ within this wedge which will enforce a single-valued solution everywhere except on the shock.

In order to derive an expression for the shock, we follow the method presented in class on 2/17/06. Recall that the primary property of a shock is that it preserves the area $A(y)$ of the waveform. We can express the area of the solution $u(x, y)$ as follows.

$$A(y) = \int_{-\infty}^{s(y)} u(x, y)dx + \int_{s(y)}^{\infty} u(x, y)dx$$

We can express the constraint that the area doesn't change as

$$\frac{dA(y)}{dy} = \int_{-\infty}^{s(y)} u_y(x, y)dx + \int_{s(y)}^{\infty} u_y(x, y)dx + [u_-(s(y), y) - u_+(s(y), y)] \frac{ds}{dy} = 0$$

Now let us consider the first integral. From the PDE we know that $u_y = -(1 + ku)u_x$.

$$I_- = -\int_{-\infty}^{s(y)} u_y(x, y)dx = \int_{-\infty}^{s(y)} (1 + ku)u_x dx = \int_{-\infty}^{s(y)} \frac{\partial}{\partial x}(u + \frac{k}{2}u^2)dx$$

$$\Rightarrow I_- = u + \frac{k}{2}u^2 \Big|_{x=-\infty}^{x=s(y)} \equiv A(u_-)$$

A similar result holds for the second integral. As a result, we arrive at the so-called "jump condition"

$$A(u_+) - A(u_-) = (u_+ - u_-)\frac{ds}{dy} \tag{8}$$

For this specific problem, we have $A(u_+) = 0$, $u_+ = 0$, and $u_- = 1$. Substituting into Equation 8, we find

$$\frac{ds}{dy} = A(u_-) = u_- + \frac{k}{2}u_-^2 = 1 + \frac{k}{2}$$

We can integrate this ODE to obtain the solution for the shock

$$s(y) = \left(1 + \frac{k}{2}\right) y + C, \text{ for } C \in \mathbb{R}$$

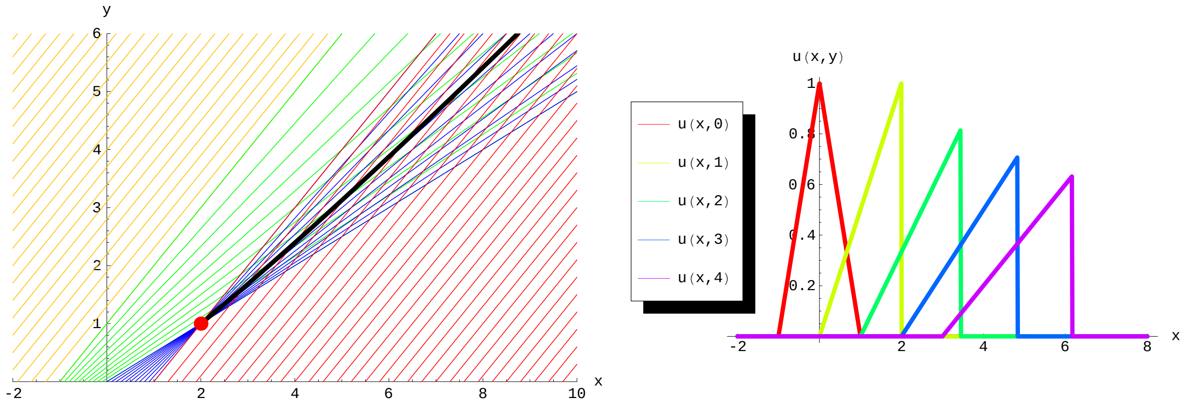
Using the wave-breaking point $(x, y) = (0, 0) \Rightarrow s(0) = 0$, we find $C = 0$ and the shock is given by

$$\boxed{\text{shock } \mathcal{S} : s(y) = \left(1 + \frac{k}{2}\right) y, \text{ for } y \geq 0} \tag{9}$$

The shock $s(y)$ is shown as a black line is Figure 3(a).

Given the closed-form expression $s(y)$ for the shock, we can define the weak solution using the solutions for each region.

$$\boxed{u(x, y) = \begin{cases} 0, & s(y) < x \leq \infty \\ 1, & -\infty \leq x < s(y) \end{cases}, \text{ for } -\infty \leq x \leq \infty, y \geq 0} \tag{10}$$




(a) characteristics, breakpoint, and shock (b) plot of weak solution $u(x,y)$ for several times $y$

Figure 4: Characteristics and weak solution found in Problem 4 (for $k=1$).

# Problem 4

Consider $u_0(x)$, for $k=1$, given by

$$u_0(x) = \begin{cases} 0, & x \geq 1 \\ -x+1, & 0 < x < 1 \\ x+1, & -1 < x \leq 0 \\ 0, & x \leq -1 \end{cases}$$

The four regions of $u_0(x)$ will give rise to differing characteristics and solutions. As a result, we proceed as before and evaluate the behavior within each region.

**Region I:** $-\infty \leq \eta \leq -1$, $u_0(\eta) = 0$

Substituting $u_0(\eta) = 0$ into Equation 4 (for $k=1$), we find that the characteristics are given by

$$x = y + \eta \Rightarrow \eta = x - y$$

Similarly, the solution within Region I must be

$$u_{\mathrm{I}}(x,y) = 0, \text{ for } -\infty \leq x - y \leq -1$$

Note that the characteristics for Region I are sketched in orange in Figure 4(a).

**Region II:** $-1 < \eta \leq 0$, $u_0(\eta) = \eta + 1$

Substituting $u_0(\eta) = \eta + 1$ into Equation 4 (for $k=1$), we find that the characteristics are given by

$$x = [1 + (\eta + 1)]y + \eta \Rightarrow \eta = \frac{x - 2y}{y+1}$$

Similarly, the solution within Region II must be

$$u_{\mathrm{II}}(x,y) = \frac{x-y+1}{y+1}, \text{ for } -1 < \frac{x-2y}{y+1} \leq 0$$

Note that the characteristics for Region II are sketched in green in Figure 4(a).

**Region III:** $0 < \eta < 1$, $u_0(\eta) = 1 - \eta$

Substituting $u_0(\eta) = 1 - \eta$ into Equation 4 (for $k = 1$), we find that the characteristics are given by

$$x = [1 + (1 - \eta)]y + \eta \Rightarrow \eta = \frac{x - 2y}{1 - y}$$

Similarly, the solution within Region III must be

$$u_{\mathrm{III}}(x, y) = \frac{x - y - 1}{y - 1}, \text{ for } 0 < \frac{x - 2y}{1 - y} < 1$$

Note that the characteristics for Region III are sketched in blue in Figure 4(a). Also notice that the characteristics for Region II intersect those in Region III – indicating the presence of the wave-breaking phenomenon.

**Region IV:** $1 \leq \eta \leq \infty$, $u_0(\eta) = 0$

Substituting $u_0(\eta) = 0$ into Equation 4 (for $k = 1$), we find that the characteristics are given by

$$x = y + \eta \Rightarrow \eta = x - y$$

Similarly, the solution within Region IV must be

$$u_{\mathrm{IV}}(x, y) = 0, \text{ for } 1 \leq x - y \leq \infty$$

Note that the characteristics for Region IV are sketched in red in Figure 4(a). Also notice that the characteristics for Region II and Region III intersect with those in Region IV – further indicating the presence of the wave-breaking phenomenon.

From Figure 4(a), it is apparent that the solutions for Regions II, III, and IV are all valid within the wedge given by $y + 1 < x < 2y$, $y \geq 1$. As a result, we must introduce a shock $\mathcal{S}$ within this wedge which will enforce a single-valued solution everywhere except on the shock. In order to derive an expression for the shock, we follow the method presented in Problem 3.

First, we'd like to determine the position and time at which wave-breaking occurs. Recall from class on 2/22/06 that, for $y > 0$, the "crest" of the wave will propagate with velocity $1 + k = 2$ for $k = 1$, whereas the "front" of the wave will propagate with unit velocity 1. As a result, the "crest" of the wave will be located at $x_{\text{crest}}(y) = 2y$, whereas the "front" will be located at $x_{\text{front}}(y) = y + 1$. A vertical tangent, corresponding to the point of wave-breaking, will appear when these two positions are coincident.

$$\Rightarrow 2y = y + 1 \Rightarrow y = 1$$

As a result, the initial time of wave-breaking will be $y = 1$ and the position will be $x = 2$.

$$\boxed{\text{wave-breaking point: } x = 2, y = 1} \tag{11}$$

Note that this point is shown as a red circle in Figure 4(a). In addition, it is located at the point where the characteristics first intersect – as typical for the wave-breaking point.

Now we turn our attention to solving for the shock $\mathcal{S}$. Recall that the "jump condition" is given by Equation 8

$$A(u_+) - A(u_-) = (u_+ - u_-)\frac{ds}{dy}$$

For this example, we have $A(u_+) = 0$ and $u_+ = 0$. Substituting into the previous equation, we find

$$A(u_-) = u_-\frac{ds}{dy}$$

Recall from Problem 3 that $A(u_-) = u_-(1 + \frac{k}{2}u_-)$. As a result, we have

$$u_-\frac{ds}{dy} = u_-(1 + \frac{k}{2}u_-) \Rightarrow \frac{ds}{dy} = 1 + \frac{k}{2}u_-$$

In this case, we want $u_- = u_{\mathrm{II}}(x, y)$ such that

$$\frac{ds}{dy} = 1 + \frac{k}{2}\left(\frac{x - y + 1}{y + 1}\right)$$

Given that $x = s(y)$ on the shock $\mathcal{S}$, then we must solve the following ODE for the shock (subject to the initial condition s(1) = 2 given by the wave-breaking point)

$$\frac{ds}{dy} = 1 + \frac{k}{2}\left(\frac{s(y) - y + 1}{y + 1}\right), s(1) = 2$$

As suggested in class, I used *Mathematica* to solve this ODE (see attached notebook at the end of this write-up). In conclusion, the shock is given by

$$\boxed{\text{shock } \mathcal{S} : s(y) = y - 1 + \sqrt{2(y + 1)}, \text{ for } y \geq 1} \tag{12}$$

Given the closed-form expression $s(y)$ for the shock, we can define the weak solution using the solutions for each region. Prior to the wave-breaking at $y = 1$, we have the solution

$$\boxed{u(x, y) = \begin{cases} 0, & 1 \leq x - y \leq \infty \\ \frac{x-y-1}{y-1}, & 0 < \frac{x-2y}{1-y} < 1 \\ \frac{x-y+1}{y+1}, & -1 < \frac{x-2y}{y+1} \leq 0 \\ 0, & -\infty \leq x - y \leq -1 \end{cases}, \text{ for } -\infty \leq x \leq \infty, 0 \leq y < 1} \tag{13}$$

After the point of wave-breaking, we have the weak solution given by

$$\boxed{u(x, y) = \begin{cases} 0, & s(y) < x \leq \infty \\ \frac{x-y+1}{y+1}, & y - 1 < x < s(y) \\ 0, & -\infty \leq x \leq y - 1 \end{cases}, \text{ for } -\infty \leq x \leq \infty, y \geq 1} \tag{14}$$

The solution $u(x, y)$ is sketched for several times $y$ in Figure 4(b).

# References

[1] Prof. J. A. Blume. Solution Set 3. http://www.engin.brown.edu/courses/en202/homework/hw3/hw3s.pdf.

[2] Douglas R. Lanman. Problem Set 3. http://mesh.brown.edu/dlanman/courses/en202/HW3.pdf.

[3] Eric W. Weisstein. Heaviside step function. http://mathworld.wolfram.com/HeavisideStepFunction.html.