EN 257: Applied Stochastic Processes
Problem Set 4

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Problem 6.12

Let the probability space $(\Omega, \mathcal{F}, P)$ be given as follows:

$$\Omega = \{a, b, c\},$$
$$\mathcal{F} = \text{ all subsets of } \Omega,$$
$$P[\{\zeta\}] = 1/3 \text{ for each outcome } \zeta.$$

Let the random sequence $X[n]$ be defined as follows:

$$X[n, a] = 3\delta[n],$$
$$X[n, b] = u[n-1],$$
$$X[n, c] = \cos(\pi n/2).$$

(a) Find the mean function $\mu_X[n]$.
(b) Find the correlation function $R_{XX}[m, n]$.
(c) Are $X[1]$ and $X[0]$ independent? Why?

Part (a)

Recall, from page 319 in [4], that the mean function of a random sequence $X[n]$ is given by

$$\mu_X[n] \triangleq E\{X[n]\} = \sum_{i=1}^{m} x_i P\{X[n] = x_i\}, \tag{1}$$

where we have assumed for this problem that $X[n]$ is a discrete random variable that takes on the values $\{x_i\}$, for $i = 1, \ldots, m$. From the problem statement we observe that the mean function $\mu_X[n]$ can be written as the sum of two periodic functions (i.e., one defined for $n \geq 1$, another for $n < 0$, and a unique value at the origin $n = 0$). This observation will be made more concrete shortly; first, let's begin by determining the value of $\mu_X[0]$. From the problem statement we have the following values for $X[0, \zeta]$ with $\zeta \in \{a, b, c\}$.

$$X[0, a] = 3$$
$$X[0, b] = 0$$
$$X[0, c] = 1$$

Since the simple events $\{a, b, c\}$ are mutually exclusive and have equal probability $P[\{\zeta\}] = 1/3$, then we conclude that $P\{X[0] = x\}$ is given by

$$P\{X[0] = x\} = \begin{cases} 1/3, & \text{for } x = \{0, 1, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

Substituting into Equation 1, we find that the mean function has the following value at $n = 0$.

$$\mu_X[0] = \sum_{i=1}^{3} x_i P\{X[0] = x_i\} = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = \frac{4}{3}$$

By observation we conclude that there are six unique cases (i.e., probability mass functions $P\{X[n] = x_i\}$ for $\mu_X[n]$). The following table summarizes the derivation and domain of each. (Note that Case 2 and Case 7 actually describe the same underlying distribution).

Case	Domain	$X[n,\zeta]$	$P_X(x) = P\{X[n] = x_i\}$	$\mu_X[n]$
1	$n = \{0\}$	$X[n,a] = 3$ $X[n,b] = 0$ $X[n,c] = 1$	$P_X(x) = \begin{cases} 1/3, & x = \{0,1,3\} \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = \frac{4}{3}$
2	$n = \{1, 3, 5, \ldots\}$	$X[n,a] = 0$ $X[n,b] = 1$ $X[n,c] = 0$	$P_X(x) = \begin{cases} 2/3, & x = 0 \\ 1/3, & x = 1 \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = \frac{1}{3}$
3	$n = \{2, 6, 10, \ldots\}$	$X[n,a] = 0$ $X[n,b] = 1$ $X[n,c] = -1$	$P_X(x) = \begin{cases} 1/3, & x = \{-1,1,0\} \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = 0$
4	$n = \{4, 8, 12, \ldots\}$	$X[n,a] = 0$ $X[n,b] = 1$ $X[n,c] = 1$	$P_X(x) = \begin{cases} 2/3, & x = 1 \\ 1/3, & x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = \frac{2}{3}$
5	$n = \{\ldots, -5, -3, -1\}$	$X[n,a] = 0$ $X[n,b] = 0$ $X[n,c] = 0$	$P_X(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = 0$
6	$n = \{\ldots, -10, -6, -2\}$	$X[n,a] = 0$ $X[n,b] = 0$ $X[n,c] = -1$	$P_X(x) = \begin{cases} 2/3, & x = 0 \\ 1/3, & x = -1 \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = -\frac{1}{3}$
7	$n = \{\ldots, -12, -8, -4\}$	$X[n,a] = 0$ $X[n,b] = 0$ $X[n,c] = 1$	$P_X(x) = \begin{cases} 2/3, & x = 0 \\ 1/3, & x = 1 \\ 0, & \text{otherwise} \end{cases}$	$\mu_X[n] = \frac{1}{3}$

From the tabulated cases we conclude that the mean function $\mu_X[n]$ has the following solution.

$$\boxed{\mu_X[n] = \begin{cases} 4/3, & \text{for } n = 0 \\ 2/3, & \text{for } \{n \geq 4 \text{ and } \mathrm{mod}(n,4) = 0\} \\ 1/3, & \text{for } \{n \geq 1 \text{ and } \mathrm{mod}(n+1,2) = 0\} \text{ or } \{n \leq -4 \text{ and } \mathrm{mod}(n,4) = 0\} \\ 0, & \text{for } \{n \geq 2 \text{ and } \mathrm{mod}(n+2,4) = 0\} \text{ or } \{n \leq -1 \text{ and } \mathrm{mod}(n-1,2) = 0\} \\ -1/3, & \text{for } \{n \leq -2 \text{ and } \mathrm{mod}(n-2,4) = 0\} \end{cases}}$$

Note that, for generality, we have expressed the mean function in terms of the common residue of $m (\mathrm{mod}\, n) \triangleq \mathrm{mod}(m,n)$.

Part (b)

We begin by reviewing the basic properties of the autocorrelation function $R_{XX}[m,n]$ and autocovariance function $K_{XX}[m,n]$, as defined for the discrete-valued random sequence $X[n]$. Recall, from pages 319 and 320 in [4], that $R_{XX}[m,n]$ and $K_{XX}[m,n]$ have the following forms for the random sequence $X[n]$.

$$R_{XX}[m,n] \triangleq E\{X[m]X^*[n]\}$$

$$K_{XX}[m,n] \triangleq E\{(X[m] - \mu_X[m])(X[n] - \mu_X[n])^*\} \tag{2}$$

In addition, we recall the following familiar relationship between the two functions.

$$R_{XX}[m,n] = K_{XX}[m,n] + \mu_X[m]\mu_X^*[n] \tag{3}$$

Since $\mu_X[n]$ is not constant, we conclude that $X[n]$ is not a stationary random sequence. As a result, we do not expect the autocorrelation function $R_{XX}[m,n]$ to be shift-invariant. Due to this complexity, we elect to evaluate the autocovariance function $K_{XX}[m,n]$ instead and apply Equation 3 to obtain $R_{XX}[m,n]$ using the previously-determined mean function. As we'll argue in Part (c), since $X[m]$ and $X[n]$ are independent for $m \neq n$, the autocovariance matrix will be diagonal such that

$$K_{XX}[m,n] = \sigma_X^2[n]\delta[m-n] = \begin{cases} E\{|X[n] - \mu_X[n]|^2\}, & \text{for } m = n \\ 0, & \text{for } m \neq n \end{cases},$$

where $\delta[m-n]$ is the discrete-time impulse and $\sigma_X^2[n]$ is the variance of the random variable $X[n]$. Using the probability mass functions tabulated in Part (a), we can evaluate $\sigma_X^2[n]$ using the following expression.

$$\sigma_X^2[n] = E\{|X[n] - \mu_X[n]|^2\} = \sum_{i=1}^{m} |x_i - \mu_X[n]|^2 P\{X[n] = x_i\}$$

This leads to the following expression for the autocorrelation function $R_{XX}[m,n]$ in terms of the mean function $\mu_X[n]$ and the variance function $\sigma_X^2[n]$.

$$\boxed{\begin{gathered} R_{XX}[m,n] = \sigma_X^2[n]\delta[m-n] + \mu_X[m]\mu_X[n] \\ \sigma_X^2[n] = \begin{cases} 14/9, & \text{for } n = 0 \\ 2/3, & \text{for } \{n \geq 2 \text{ and } \mathrm{mod}(n+2,4) = 0\} \\ 2/9, & \text{for } \{n \geq 1 \text{ and } \mathrm{mod}(n+2,4) \neq 0\} \text{ or } \{n \leq -2 \text{ and } \mathrm{mod}(n,2) = 0\} \\ 0, & \text{for } \{n \leq -1 \text{ and } \mathrm{mod}(n-1,2) = 0\} \end{cases} \end{gathered}}$$

Part (c)

We begin by recalling that it is a necessary, but not sufficient condition, that the covariance $K_{XX}[m,n]$ be equal to zero if two random variables $X[n]$ and $X[m]$ are independent. Using the results from Part (b), we conclude that $K_{XX}[0,1] = 0$; as a result, we cannot deny that $X[0]$ and $X[1]$ are dependent based solely on their correlation. (Neither can we conclude that they are necessarily independent.) To resolve this ambiguity we note that the random variables $X[0]$ and $X[1]$ must be independent, since they are generated by independent random processes. That is, to generate a sample of $X[0]$, we first must uniformly select an event $\zeta \in \{a,b,c\}$. Afterwards, we will evaluate the deterministic function corresponding to each event. A simple, and most-importantly independent, selection procedure must be applied to generate a sample of $X[1]$. Since the sampling occurs independently, knowledge of $X[0]$ does not alter out expectation of the value of $X[1]$. As a result we conclude that $X[0]$ and $X[1]$ are independent random variables.

Problem 6.13

Consider a random sequence $X[n]$ as the input to a linear filter with impulse response

$$h[n] = \begin{cases} 1/2, & n = \{0, 1\} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Denote the output random sequence $Y[n]$, for each outcome ζ, as

$$Y[n, \zeta] = \sum_{k=-\infty}^{\infty} h[k] X[n-k, \zeta].$$

Assume the filter runs for all time (i.e., $-\infty < n < \infty$) and that we are given the mean function of the input $\mu_X[n]$ and the autocorrelation function of the input $R_{XX}[n_1, n_2]$.

(a) Find the mean function of the output $\mu_Y[n]$.
(b) Find the output autocorrelation function $R_{YY}[n_1, n_2]$.
(c) Write the output autocovariance function $K_{YY}[n_1, n_2]$ using answers from parts (a) and (b).
(d) Now assume that the input $X[n]$ is a Gaussian random sequence. Write the joint pdf of the output $f_Y(y_1, y_2; n_1, n_2)$ at two arbitrary times $n_1 \neq n_2$ in terms of $\mu_Y[n]$ and $K_{YY}[n_1, n_2]$.

Part (a)

Recall, from Section 6.3 in [4], that the output mean function $\mu_Y[n]$ can be obtained as follows.

$$\begin{aligned} \mu_Y[n] = E\{Y[n, \zeta]\} &= E\left\{ \sum_{k=-\infty}^{\infty} h[k] X[n-k, \zeta] \right\} \\ &= \sum_{k=-\infty}^{\infty} h[k] E\{X[n-k, \zeta]\} = \sum_{k=-\infty}^{\infty} h[k] \mu_X[n-k] \end{aligned} \tag{5}$$

Substituting Equation 4 into Equation 5 gives the following solution for the output mean function.

$$\boxed{\mu_Y[n] = \frac{1}{2}\left(\mu_X[n] + \mu_X[n-1]\right)} \tag{6}$$

Part (b)

Recall, from Equation 6.3-7 on page 344 in [4], that the output autocorrelation function $R_{YY}[n_1, n_2]$ for a linear system with time-variant impulse response $h[n, k]$ is given by the following expression.

$$R_{YY}[n_1, n_2] = \sum_{k=-\infty}^{\infty} h[n_1, k] \left(\sum_{l=-\infty}^{\infty} h^*[n_2, l] R_{XX}[k, l] \right)$$

Since the impulse response in Equation 4 is real-valued and shift-invariant, we can simply the previous equation as follows.

$$R_{YY}[n_1, n_2] = \sum_{k=-\infty}^{\infty} h[n_1 - k] \left(\sum_{l=-\infty}^{\infty} h[n_2 - l] R_{XX}[k, l] \right),$$

Substituting Equation 4, we find the following solution for the output autocorrelation function.

$$\boxed{R_{YY}[n_1, n_2] = \frac{1}{4}\left(R_{XX}[n_1, n_2] + R_{XX}[n_1 - 1, n_2] + R_{XX}[n_1, n_2 - 1] + R_{XX}[n_1 - 1, n_2 - 1]\right)}$$

Part (c)

This problem is similar to part (b). First we recall, from Equation 6.3-11, that the output autocovariance function $K_{YY}[n_1, n_2]$ for a linear system with time-variant impulse response $h[n, k]$ is given by the following expression.

$$K_{YY}[n_1, n_2] = \sum_{k=-\infty}^{\infty} h[n_1, k]\left(\sum_{l=-\infty}^{\infty} h^*[n_2, l]K_{XX}[k, l]\right)$$

Since the impulse response in Equation 4 is real-valued and shift-invariant, we can simply the previous equation as follows.

$$K_{YY}[n_1, n_2] = \sum_{k=-\infty}^{\infty} h[n_1 - k]\left(\sum_{l=-\infty}^{\infty} h[n_2 - l]K_{XX}[k, l]\right),$$

Substituting Equation 4, we find the following solution for the output autocovariance function $K_{YY}[n_1, n_2]$ in terms of the input autocovariance function $K_{XX}[n_1, n_2]$.

$$K_{YY}[n_1, n_2] = \frac{1}{4}\left(K_{XX}[n_1, n_2] + K_{XX}[n_1 - 1, n_2] + K_{XX}[n_1, n_2 - 1] + K_{XX}[n_1 - 1, n_2 - 1]\right)$$

To complete our derivation we need to find an expression linking the input autocovariance $K_{XX}[n_1, n_2]$ with the input mean $\mu_X[n]$ and input autocorrelation $R_{XX}[n_1, n_2]$. Recall that Equation 6.1-13 on page 320 in [4] gives precisely this relationship.

$$K_{XX}[n_1, n_2] = R_{XX}[n_1, n_2] - \mu_X[n_1]\mu_X^*[n_2]$$

Substituting this expression into the previous result gives the output autocovariance $K_{YY}[n_1, n_2]$ as a function of $\mu_X[n]$ and $R_{XX}[n_1, n_2]$.

$$\begin{aligned} K_{YY}[n_1, n_2] = \frac{1}{4}\big(&R_{XX}[n_1, n_2] + R_{XX}[n_1 - 1, n_2] + R_{XX}[n_1, n_2 - 1] + R_{XX}[n_1 - 1, n_2 - 1] - \\ &\mu_X[n_1]\mu_X^*[n_2] - \mu_X[n_1 - 1]\mu_X^*[n_2] - \mu_X[n_1]\mu_X^*[n_2 - 1] - \mu_X[n_1 - 1]\mu_X^*[n_2 - 1]\big) \end{aligned}$$

Finally, we note that the solutions to parts (a) and (b) can be substituted to simply this expression. This yields an equivalent solution for the output autocovariance function $K_{YY}[n_1, n_2]$ in terms of $\mu_Y[n]$ and $R_{YY}[n_1, n_2]$.

$$\boxed{K_{YY}[n_1, n_2] = R_{YY}[n_1, n_2] - \mu_Y[n_1]\mu_Y^*[n_2]}$$

Part (d)

Recall, from Definition 6.1-3 on page 323 in [4], that a random sequence $X[n]$ is Gaussian if its N^{th}-order distribution functions are jointly Gaussian for all $N \geq 1$. In other words, if $X[n]$ is a

Gaussian random sequence, then we can express the joint pdf of the input $f_X(x_1, x_2; n_1, n_2)$ at two arbitrary times $n_1 \neq n_2$ as

$$f_X(x_1, x_2; n_1, n_2) = \frac{1}{2\pi|\mathbf{K}_X|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu_X)^T \mathbf{K}_X^{-1}(\mathbf{x} - \mu_X)\right),$$

where the parameters $\mathbf{x}$, μ_X, and $\mathbf{K}_X$ are given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \mu_X = \begin{pmatrix} \mu_X[n_1] \\ \mu_X[n_2] \end{pmatrix}, \text{ and } \mathbf{K}_X = \begin{pmatrix} K_{XX}[n_1, n_1] & K_{XX}[n_1, n_2] \\ K_{XX}[n_2, n_1] & K_{XX}[n_2, n_2] \end{pmatrix}.$$

At this point we recall the important fact, as given by Theorem 5.6-1, that the linear transformation of a Gaussian random vector produces another Gaussian random vector. As a result, the output random sequence $Y[n]$ will also be a Gaussian random sequence with the following joint pdf.

$$\boxed{\begin{array}{c} f_Y(y_1, y_2; n_1, n_2) = \frac{1}{2\pi|\mathbf{K}_Y|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mu_Y)^T \mathbf{K}_Y^{-1}(\mathbf{y} - \mu_Y)\right), \text{ where} \\ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \mu_Y = \begin{pmatrix} \mu_Y[n_1] \\ \mu_Y[n_2] \end{pmatrix}, \text{ and } \mathbf{K}_Y = \begin{pmatrix} K_{YY}[n_1, n_1] & K_{YY}[n_1, n_2] \\ K_{YY}[n_2, n_1] & K_{YY}[n_2, n_2] \end{pmatrix} \end{array}}$$

Problem 6.22

Let $W[n]$ be an independent random sequence with constant mean $\mu_W = 0$ and variance σ_W^2. Define a new random sequence $X[n]$ as follows:

$$\begin{aligned} X[0] &= 0 \\ X[n] &= \rho X[n-1] + W[n] \text{ for } n \geq 1. \end{aligned}$$

(a) Find the mean value of $X[n]$ for $n \geq 0$.

(b) Find the autocovariance of $X[n]$, denoted as $K_{XX}[m,n]$.

(c) For what values of ρ does $K_{XX}[m,n]$ tend to $G[m-n]$, for some finite-valued function G, as m and n become large? (This situation is known as *asymptotic stationarity*.)

Part (a)

Let's begin by determining the general form for $X[n]$. Following the derivation presented in class, we can evaluate the first few terms in the sequence directly.

$$\begin{aligned} X[1] &= \rho X[0] + W[1] \\ X[2] &= \rho(\rho X[0] + W[1]) + W[2] = \rho^2 X[0] + \rho W[1] + W[2] \\ X[3] &= \rho(\rho^2 X[0] + \rho W[1] + W[2]) + W[3] = \rho^3 X[0] + \rho^2 W[1] + \rho W[2] + W[3] \end{aligned}$$

By inspection, we conclude that the general form for $X[n]$ is given by

$$X[n] = \rho^n X[0] + \sum_{m=1}^{n} \rho^{n-m} W[m],$$

where $\rho^n X[0]$ is the homogeneous solution to $X[n] = \rho X[n-1]$. Substituting the initial condition $X[0] = 0$ yields the specific solution for $X[n]$.

$$X[n] = \sum_{m=1}^{n} \rho^{n-m} W[m] \tag{7}$$

At this point we recall, from page 319 in [4], that the mean function of a random sequence is given by the following expression.

$$\mu_X[n] \triangleq E\{X[n]\}$$

Substituting Equation 7 and exploiting the linearity of the expectation operator, we find

$$\mu_X[n] = E\left\{\sum_{m=1}^{n} \rho^{n-m} W[m]\right\} = \sum_{m=1}^{n} \rho^{n-m} E\{W[m]\} = \sum_{m=1}^{n} \rho^{n-m} \mu_W = 0.$$

As a result, we conclude that the random sequence $X[n]$ is mean-zero for all $n \geq 0$.

$$\boxed{\mu_X[n] = \mu_X = 0, \text{ for } n \geq 0}$$

Part (b)

Recall, from Equation 6.1-10, that the autocovariance $K_{XX}[m,n]$ is defined as follows.

$$K_{XX}[m,n] \triangleq E\{(X[m]-\mu_X[m])(X[n]-\mu_X[n])^*\}$$

Substituting Equation 7 and the result $\mu_X = 0$, we obtain the following expression for $K_{XX}[m,n]$.

$$\begin{aligned} K_{XX}[m,n] &= E\left\{\left(\sum_{i=1}^{m}\rho^{m-i}W[i]\right)\left(\sum_{j=1}^{n}\rho^{n-j}W[j]\right)^*\right\} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\rho^{m-i}(\rho^*)^{n-j}E\left\{W[i]W^*[j]\right\} \end{aligned} \tag{8}$$

At this point, we recall that the variance $\sigma_W^2[n]$ of $W[n]$ is given by the following expression.

$$\sigma_W^2[n] = Var\left\{W[n]\right\} \triangleq E\left\{(W[n]-\mu_W[n])(W[n]-\mu_W[n])^*\right\}$$

Since $\mu_W[n] = 0$, we have

$$\sigma_W^2[n] = \sigma_W^2 = E\left\{W[n]W^*[n]\right\}, \text{ for } n \geq 0.$$

In addition, we recall from Definition 6.1-2 that an independent random sequence is one whose random variables at any times $\{n_1, n_2, \ldots, n_N\}$ are jointly independent for all positive integers N. As a result, we conclude that $E\left\{W[m]W^*[n]\right\}$ is given by the following expression.

$$E\left\{W[m]W^*[n]\right\} = \begin{cases} \sigma_W^2, & \text{for } m = n \\ 0, & \text{otherwise} \end{cases}$$

Substituting this result into Equation 8 gives the following expression for $K_{XX}[m,n]$.

$$K_{XX}[m,n] = \begin{cases} \sum_{i=1}^{n}\rho^{m-i}(\rho^*)^{n-i}\sigma_W^2, & \text{for } m \geq n \\ \sum_{i=1}^{m}\rho^{m-i}(\rho^*)^{n-i}\sigma_W^2, & \text{for } m < n \end{cases}$$

Following the derivation in class, we conclude that these geometric series will converge for $|\rho| < 1$, such that the solution for $K_{XX}[m,n]$ is given by the following expression.

$$\boxed{K_{XX}[m,n] = \begin{cases} \left[\frac{\rho^{m-n}(1-|\rho|^{2n})}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m \geq n \\ \left[\frac{(\rho^*)^{n-m}(1-|\rho|^{2m})}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m < n \end{cases}, \text{ for } |\rho| < 1}$$

As an aside, we note that $|\rho| < 1$ is a reasonable assumption, since this ensures bounded-input/bounded-output (BIBO) stability. Also, for $\rho \in \mathbb{R}$, this solution reduces to that found in class.

Part (c)

Finally, we conclude by noticing that $X[n]$ is asymptotically stationary for $|\rho| < 1$. That is, in the limit that m and n become large, $K_{XX}[m,n]$ is only a function of the time shift $m-n$ such that

$$\lim_{m\to\infty,\ n\to\infty} K_{XX}[m,n] = G[m-n] = \begin{cases} \left[\frac{\rho^{m-n}}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m \geq n \\ \left[\frac{(\rho^*)^{n-m}}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m < n \end{cases}, \text{ for } |\rho| < 1$$

Problem 9.3

Use the orthogonality principle to show that the minimum mean-square error (MMSE)

$$\varepsilon^2 \triangleq E[(X - E[X|Y])^2], \tag{9}$$

for real-valued random variables, can be expressed as

$$\varepsilon^2 = E[X(X - E[X|Y])]$$

or as

$$\varepsilon^2 = E[X^2] - E[E[X|Y]^2].$$

Generalize to the case where $\mathbf{X}$ and $\mathbf{Y}$ are real-valued random vectors. That is, show that the MMSE matrix is

$$\begin{aligned}\varepsilon^2 &\triangleq E[(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] \\ &= E[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] \\ &= E[\mathbf{X}\mathbf{X}^T] - E[E[\mathbf{X}|\mathbf{Y}]E^T[\mathbf{X}|\mathbf{Y}]].\end{aligned} \tag{10}$$

Let's begin by expanding the product in Equation 9.

$$\begin{aligned}\varepsilon^2 &= E[(X - E[X|Y])(X - E[X|Y])] \\ &= E[X(X - E[X|Y]) - E[X|Y](X - E[X|Y])] \\ &= E[X(X - E[X|Y])] - E[E[X|Y](X - E[X|Y])]\end{aligned} \tag{11}$$

At this point we recall the *orthogonality principle*, as given by Property 9.1-1 on page 555 in [4] and Theorem 5.4.1 on page 327 in [2]. That is, the MMSE error vector

$$\varepsilon \triangleq \mathbf{X} - E[\mathbf{X}|\mathbf{Y}]$$

is orthogonal to any measurable function $h(\mathbf{Y})$ of the data, such that

$$E[h^*(\mathbf{Y})(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] = \mathbf{0}. \tag{12}$$

For the random variables X and Y, Equation 12 yields the following condition for $h^*(Y) \triangleq E[X|Y]$.

$$E[E[X|Y](X - E[X|Y])] = 0$$

Substituting this result into Equation 11 yields the desired relation via the orthogonality principle.

$$\varepsilon^2 = E[X(X - E[X|Y])]$$

To complete the scalar-valued derivation, we further expand this product as follows.

$$\varepsilon^2 = E[X(X - E[X|Y])] = E[X^2] - E[XE[X|Y]]$$

Recall, from Equation 4.2-27 in [4], the smoothing property of the conditional expectation ensures

$$E[X] = E[E[X|Y]]$$

for the random variables X and Y. Applying this condition to the previous expression yields the final solution.

$$\therefore \boxed{\varepsilon^2 = E[X(X - E[X|Y])] = E[X^2] - E[E[X|Y]^2]}$$

Now let's generalize to the case where $\mathbf{X}$ and $\mathbf{Y}$ are real-valued random vectors. We begin by expanding the product in Equation 10.

$$\begin{aligned}
\varepsilon^2 &= E[(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] \\
&= E[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T - E[\mathbf{X}|\mathbf{Y}](\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] \\
&= E[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] - E[E[\mathbf{X}|\mathbf{Y}](\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] \qquad (13)
\end{aligned}$$

For the random vectors $\mathbf{X}$ and $\mathbf{Y}$, Equation 12 yields the following condition for $h^*(\mathbf{Y}) \triangleq E[\mathbf{X}|\mathbf{Y}]$.

$$E[E[\mathbf{X}|\mathbf{Y}](\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] = \mathbf{0}$$

Substituting this result into Equation 13 yields the desired relation via the orthogonality principle.

$$\varepsilon^2 = E[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T]$$

To complete the vector-valued derivation, we further expand this product as follows.

$$\varepsilon^2 = E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}E^T[\mathbf{X}|\mathbf{Y}]]$$

As in the scalar-valued case, the smoothing property of the conditional expectation ensures

$$E[\mathbf{X}] = E[E[\mathbf{X}|\mathbf{Y}]].$$

Applying this condition to the previous expression yields the desired solution.

$$\therefore \boxed{\varepsilon^2 = E[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T] = E[\mathbf{X}\mathbf{X}^T] - E[E[\mathbf{X}|\mathbf{Y}]E^T[\mathbf{X}|\mathbf{Y}]]}$$

(QED)

Problem 9.8

A random sequence $Y[n]$, for $n = 0, 1, 2, \ldots$, satisfies the second-order linear difference equation

$$2Y[n+2] + Y[n+1] + Y[n] = 2W[n], \text{ for } Y[0] = 0, Y[1] = 1,$$

with $W[n]$ a standard white Gaussian random sequence. Transform this equation into the state-space representation and evaluate the mean function $\mu_{\mathbf{X}}[n]$ and the correlation function $\mathbf{R}_{\mathbf{XX}}[n_1, n_2]$ for at least the first few values of n. (Hint: Define the state vector $\mathbf{X}[n] \triangleq (Y[n+2], Y[n+1])^T$.)

As requested, let's begin by transforming the linear constant coefficient difference equation into the state-space representation. Following the method outlined in Example 6.6-2 on page 374 in [4], we conclude that the state-space representation has the following form.

$$\boxed{\begin{gathered}\mathbf{X}[n] = \mathbf{A}\mathbf{X}[n-1] + \mathbf{b}W[n], \text{ where} \\ \mathbf{X}[n] = \begin{pmatrix} Y[n+2] \\ Y[n+1] \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -1/2 & -1/2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{X}[-1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{gathered}}$$

To confirm this expression, we write out the matrix-vector product and compare to the original difference equation.

$$\begin{pmatrix} Y[n+2] \\ Y[n+1] \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y[n+1] \\ Y[n] \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} W[n]$$

Now recall that the general solution to the resulting vector-valued difference equation is given by Equation 9.2-2 on page 571 in [4].

$$\mathbf{X}[n] = \mathbf{A}^{n+1}\mathbf{X}[-1] + \sum_{m=0}^{n} \mathbf{A}^{n-m}\mathbf{b}W[m]$$

The mean function can be obtained using the standard definition as follows.

$$\begin{aligned}\mu_{\mathbf{X}}[n] = E\{\mathbf{X}[n]\} &= E\left\{\mathbf{A}^{n+1}\mathbf{X}[-1] + \sum_{m=0}^{n} \mathbf{A}^{n-m}\mathbf{b}W[m]\right\} \\ &= \mathbf{A}^{n+1}\mathbf{X}[-1] + \sum_{m=0}^{n} \mathbf{A}^{n-m}\mathbf{b}E\{W[m]\} \\ &= \mathbf{A}^{n+1}\mathbf{X}[-1]\end{aligned}$$

Note that in the previous expression we have exploited the linearity property of the expectation operator and the fact that $E\{W[m]\} = 0, \ \forall n$. As a result, we conclude that the mean function $\mu_{\mathbf{X}}[n]$ is given by the following expression (with the resulting first few values also shown below).

$$\boxed{\begin{gathered}\mu_{\mathbf{X}}[n] = \mathbf{A}^{n+1}\mathbf{X}[-1] \\ \mu_{\mathbf{X}}[0] = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \quad \mu_{\mathbf{X}}[1] = \begin{pmatrix} -1/4 \\ -1/2 \end{pmatrix}, \quad \text{and} \quad \mu_{\mathbf{X}}[2] = \begin{pmatrix} 3/8 \\ -1/4 \end{pmatrix}\end{gathered}}$$

To complete our analysis, we recall that the autocorrelation function $\mathbf{R}_{\mathbf{XX}}[n_1, n_2]$ can also be obtained using the standard definition

$$\mathbf{R}_{\mathbf{XX}}[n_1, n_2] = E\{\mathbf{X}[n_1]\mathbf{X}^\dagger[n_2]\},$$

where $\mathbf{X}^\dagger[n]$ denotes the conjugate transpose of $\mathbf{X}[n]$. In this problem $\mathbf{X}$ is real-valued, so we conclude that $\mathbf{R_{XX}}[n_1, n_2]$ is given by the following expression.

$$\mathbf{R_{XX}}[n_1, n_2] = E\{\mathbf{X}[n_1]\mathbf{X}^T[n_2]\}$$

Substituting the general solution for $\mathbf{X}[n]$, we find the following result.

$$\begin{aligned}
\mathbf{R_{XX}}[n_1, n_2] &= E\left\{\left(\mathbf{A}^{n_1+1}\mathbf{X}[-1] + \sum_{m_1=0}^{n_1} \mathbf{A}^{n_1-m_1}\mathbf{b}W[m_1]\right)\left(\mathbf{A}^{n_2+1}\mathbf{X}[-1] + \sum_{m_2=0}^{n_2} \mathbf{A}^{n_2-m_2}\mathbf{b}W[m_2]\right)^T\right\} \\
&= E\left\{\left(\mathbf{A}^{n_1+1}\mathbf{X}[-1] + \sum_{m_1=0}^{n_1} \mathbf{A}^{n_1-m_1}\mathbf{b}W[m_1]\right)\left(\mathbf{X}^T[-1](\mathbf{A}^T)^{n_2+1} + \sum_{m_2=0}^{n_2} \mathbf{b}^T(\mathbf{A}^T)^{n_2-m_2}W[m_2]\right)\right\}
\end{aligned}$$

Once again we can exploit the linearity property of the expectation operator. In addition, notice that the cross-terms in $W[n]$ will be eliminated since $E\{W[n]\} = 0,\ \forall n$. As a result, the autocorrelation function has the following solution.

$$\mathbf{R_{XX}}[n_1, n_2] = \mathbf{A}^{n_1+1}\mathbf{X}[-1]\mathbf{X}^T[-1](\mathbf{A}^T)^{n_2+1} + \sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2} \mathbf{A}^{n_1-m_1}\mathbf{b}\mathbf{b}^T(\mathbf{A}^T)^{n_2-m_2}E\{W[m_1]W[m_2]\}$$

At this point we recall that $E\{W[m_1]W[m_2]\} = \sigma_W^2\delta[m_1 - m_2]$ for $W[n]$ a white Gaussian random sequence. According to the problem state, $\sigma_W^2 = 1$ which leads to the following solution for the autocorrelation function $\mathbf{R_{XX}}[n_1, n_2]$ (with the resulting first few values also shown below).

$$\boxed{\begin{gathered}
\mathbf{R_{XX}}[n_1, n_2] = \begin{cases} \mathbf{A}^{n_1+1}\mathbf{X}[-1]\mathbf{X}^T[-1](\mathbf{A}^T)^{n_2+1} + \sum_{m=0}^{n_2} \mathbf{A}^{n_1-m}\mathbf{b}\mathbf{b}^T(\mathbf{A}^T)^{n_2-m}, & \text{for } n_1 \geq n_2 \\ \mathbf{A}^{n_1+1}\mathbf{X}[-1]\mathbf{X}^T[-1](\mathbf{A}^T)^{n_2+1} + \sum_{m=0}^{n_1} \mathbf{A}^{n_1-m}\mathbf{b}\mathbf{b}^T(\mathbf{A}^T)^{n_2-m}, & \text{for } n_1 < n_2 \end{cases} \\
\mathbf{R_{XX}}[0,0] = \begin{pmatrix} 5/4 & -1/2 \\ -1/2 & 1 \end{pmatrix},\ \mathbf{R_{XX}}[0,1] = \begin{pmatrix} -3/8 & 5/4 \\ -1/4 & -1/2 \end{pmatrix},\ \text{and } \mathbf{R_{XX}}[1,1] = \begin{pmatrix} 21/16 & -3/8 \\ -3/8 & 5/4 \end{pmatrix}
\end{gathered}}$$

Problem 5.4-9 [Larson and Shubert, p. 341]

Consider a population of mice in some fixed geographical area and let $X[n]$ be their number at the beginning of the n^{th} time period. Assume that during each time period each mouse present at the beginning of that period has a fixed probability p of dying, independently of all the others. Before the end of the n^{th} period, however, a random number $Y[n]$ of new mice invades the area, where $Y[n]$ is a Poisson random variable with parameter λ and is independent of $X[n]$.

(a) Find the conditional expectation $E\{X[n+1]|X[n]=x\}$ for all $n \geq 1$.

(b) Use the conditional expectation to obtain a recurrence relation for $\mu_X[n] = E\{X[n]\}$.

(c) Show the average number of mice $\mu_X[n]$ approaches a limit as $n \to \infty$ and evaluate this limit.

Part (a)

Let's begin by defining the number of mice $X[n+1]$ at the beginning of time period $n+1$. From the problem statement we have

$$X[n+1] = X[n] - D[n] + Y[n],$$

where $X[n]$ is the number of mice at the beginning of period n, $D[n]$ is the number of mice which died during the previous period, and $Y[n]$ is the number of invading mice. Since the conditional expectation operator is linear, we conclude

$$\begin{aligned} E\{X[n+1]|X[n]=x\} &= E\{X[n]|X[n]=x\} - E\{D[n]|X[n]=x\} + E\{Y[n]|X[n]=x\} \\ &= x - E\{D[n]|X[n]=x\} + E\{Y[n]|X[n]=x\}. \end{aligned} \tag{14}$$

To proceed we must determine the remaining conditional expectations in Equation 14. First, note that $D[n]$, the number of mice that died in period n, follows a binomial distribution.

$$P\{D[n]=d|X[n]=x\} = \binom{x}{d} p^d (1-p)^{x-d}$$

As a result, we conclude that the expected number of deaths is given by the following expression in $X[n]=x$ and p.

$$E\{D[n]|X[n]=x\} = \sum_{d=0}^{x} d \binom{x}{d} p^d (1-p)^{x-d} = px \tag{15}$$

Similarly, from the problem statement, we note that $Y[n]$ follows a Poisson distribution with parameter λ.

$$P\{Y[n]=y|X[n]=x\} = \frac{\lambda^y e^{-\lambda}}{y!}$$

Following Example 4.1-2 on page 173 in [4], we conclude that the expected number of invading mice is given by the following expression in $X[n]=x$ and λ.

$$E\{Y[n]|X[n]=x\} = \sum_{y=0}^{\infty} y \left(\frac{\lambda^y e^{-\lambda}}{y!} \right) = \lambda \tag{16}$$

Substituting Equations 15 and 16 into Equation 14 yields the desired expression for the conditional expectation.

$$\boxed{E\{X[n+1]|X[n]=x\} = (1-p)x + \lambda} \tag{17}$$

Part (b)

Recall from Problem 6.22 that the mean function $\mu_X[n]$ is given by

$$\mu_X[n] = E\{X[n]\}.$$

For the initial condition $X[1]$ we must have

$$\mu_X[1] = E\{X[1]\} = X[1],$$

since $X[1]$ is a known constant. By recursively applying the conditional expectation in Equation 17, we can determine the first few terms of $\mu_X[n]$.

$$\begin{aligned}\mu_X[2] &= E\{X[2]|X[1]\} = (1-p)X[1] + \lambda \\ \mu_X[3] &= E\{X[3]|X[2]\} = (1-p)^2 X[1] + (1-p)\lambda + \lambda\end{aligned}$$

By induction, we conclude that $\mu_X[n]$ is given by the following expression.

$$\mu_X[n] = \begin{cases} (1-p)^{n-1}X[1] + \lambda \sum_{i=0}^{n-2}(1-p)^i, & \text{for } n > 1 \\ X[1], & \text{for } n = 1 \end{cases}$$

For $0 < p \leq 1$ the geometric series converges and $\mu_X[n]$ has the following solution.

$$\boxed{\mu_X[n] = \begin{cases} \frac{\lambda}{p} + (1-p)^{n-1}\left(X[1] - \frac{\lambda}{p}\right), & \text{for } n > 1 \\ X[1], & \text{for } n = 1 \end{cases}} \tag{18}$$

Part (c)

For $0 < p \leq 1$ the average number of mice $\mu_X[n]$ approaches a finite limit as $n \to \infty$.

$$\lim_{n\to\infty} \mu_X[n] = \lim_{n\to\infty} \left\{ \frac{\lambda}{p} + (1-p)^{n-1}\left(X[1] - \frac{\lambda}{p}\right) \right\} = \frac{\lambda}{p}$$

Note that, since $0 < p \leq 1$, then $(1-p)^{n-1}$ tends to zero as n becomes large. As a result we conclude that, regardless of the starting population $X[1]$, the average number of mice $\mu_X[n]$ approaches the following limit as $n \to \infty$.

$$\boxed{\lim_{n\to\infty} \mu_X[n] = \frac{\lambda}{p}, \text{ for } 0 < p \leq 1}$$

Obviously, for $p = 0$, there will be no deaths and the population will grow without bound for $\lambda > 0$.

Problem 5.4-13 [Larson and Shubert, p. 342]

Let X_1 and X_2 be independent random variables both uniformly distributed on $(0, 1)$. For

$$Y = \frac{1}{2}(X_1 + X_2) \quad \text{and} \quad Z = \sqrt{X_1 X_2}$$

find the conditional expectation $E[Y|Z = z]$ for all $0 < z < 1$.

We begin our analysis by applying the linearity property of the conditional expectation operator.

$$E[Y|Z = z] = E\left[\frac{1}{2}(X_1 + X_2) \;\middle|\; Z = z\right] = \frac{1}{2}E[X_1|Z = z] + \frac{1}{2}E[X_2|Z = z] = E[X_1|Z = z] \quad (19)$$

Note that, on the right-hand side, we have used the fact that the expression is symmetric in X_1 and X_2, so we are only required to evaluate the single conditional expectation $E[X_1|Z = z]$. Now recall, from Equations 4.2-9 and 4.2-10 on page 186 in [4], that the conditional expectation of X_1 given $Z = z$ is

$$E[X_1|Z = z] \triangleq \int_{-\infty}^{\infty} x_1 f_{X_1|Z}(x_1|z) dx_1, \quad (20)$$

where the conditional probability density function is given by

$$f_{X_1|Z}(x_1|z) = \frac{f_{ZX_1}(z, x_1)}{f_Z(z)}, \text{ for } f_Z(z) \neq 0. \quad (21)$$

At this point all that remains is to determine closed-form expressions for $f_{ZX_1}(z, x_1)$ and $f_Z(z)$; substituting these expressions into Equation 21 will yield the desired solution for $E[Y|Z = z]$ via Equations 19 and 20.

The expression for $f_Z(z)$ can be found using the approach outlined in Example 3.3-1 on page 186. Specifically, we know that the following expression defines the distribution function $F_Z(z)$.

$$F_Z(z) = \int\int_{(x_1,x_2)\,\in\, C_z} f_{X_1X_2}(x_1, x_2) dx_1 dx_2, \text{ for } \{Z \leq z\} = \{(X_1, X_2) \in C_z\}$$

For X_1 and X_2 uniformly distributed on $(0, 1)$, the joint density function has the following form.

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1, & \text{for } 0 < x_1, x_2 < 1 \\ 0, & \text{otherwise} \end{cases} \quad (22)$$

To evaluate the previous expression for $F_Z(z)$, we note that X_2 can be expressed in terms of X_1 using $Z = \sqrt{X_1 X_2}$. As a result, we must have

$$x_2 = \begin{cases} \frac{z^2}{x_1}, & \text{for } z^2 \leq x_1 < 1 \\ 1, & \text{for } 0 < x_1 < z^2 \end{cases}$$

which yields the following result for the probability distribution $F_Z(z)$.

$$F_Z(z) = \int_{z^2}^{1} \left(\int_0^{\frac{z^2}{x_1}} dx_2 \right) dx_1 + \int_0^{z^2} \left(\int_0^1 dx_2 \right) dx_1 = \left[1 - \ln(z^2)\right] z^2$$

Taking the first derivative with respect to z yields the desired expression for $f_Z(z)$.

$$f_Z(z) = \frac{dF_Z(z)}{dz} = -2z\ln(z^2) \tag{23}$$

The expression for $f_{ZX_1}(z, x_1)$ can be found using the approach outlined in Example 3.5-4 on page 159. We begin by defining the pair of random variables Z and X_1 as functions of X_1 and X_2.

$$\begin{aligned} Z &\triangleq g(X_1, X_2) = \sqrt{X_1 X_2} \\ X_1 &\triangleq h(X_1, X_1) = X_1 \end{aligned}$$

Next, we observe that the equations

$$\begin{aligned} z - g(x_1, x_2) &= 0 \\ x_1 - h(x_1, x_2) &= 0 \end{aligned}$$

have only one real root, for $0 < x_1, x_2 < 1$, given by

$$\begin{aligned} x_1^1 &= \phi_1(z, x_1) = x_1 \\ x_2^1 &= \varphi_1(z, x_1) = \tfrac{z^2}{x_1}. \end{aligned} \tag{24}$$

At this point we recall that $f_{ZX_1}(z, x_1)$ can be obtained directly from $f_{X_1X_2}(x_1, x_2)$ using the methods outlined in Section 3.4 in [4]. From that section we note that the joint pdf can be expressed as

$$f_{ZX_1}(z, x_1) = \sum_{i=1}^{n} f_{X_1X_2}(x_1^i, x_2^i)|\tilde{J}_i|, \tag{25}$$

where $|\tilde{J}_i|$ is the magnitude of the Jacobian transformation such that

$$|\tilde{J}_i| = \left| \det \begin{pmatrix} \partial\phi_i/\partial z & \partial\phi_i/\partial x_1 \\ \partial\varphi_i/\partial z & \partial\varphi_i/\partial x_1 \end{pmatrix} \right| \tag{26}$$

and n is the number of solutions to the equations $z = g(x_1, x_2)$ and $x_1 = h(x_1, x_2)$. Substituting Equation 24 into Equation 26 gives the following Jacobian magnitude.

$$|\tilde{J}_1| = \left| \det \begin{pmatrix} 0 & 1 \\ \frac{2z}{x_1} & -\frac{z^2}{x_1^2} \end{pmatrix} \right| = \frac{2z}{x_1} \tag{27}$$

Substituting Equations 22, 24, and 27 into Equation 25 yields the desired expression for the joint probability density $f_{ZX_1}(z, x_1)$.

$$f_{ZX_1}(z, x_1) = \left(\frac{2z}{x_1}\right) f_{X_1X_2}\left(\frac{z^2}{x_1}, x_1\right) = \begin{cases} \frac{2z}{x_1}, & \text{for } z^2 \le x_1 < 1 \\ 0, & \text{for } 0 < x_1 < z^2 \end{cases} \tag{28}$$

Now that we have determined closed-form expressions for $f_{ZX_1}(z, x_1)$ and $f_Z(z)$, we can substitute into Equation 21 to obtain the conditional density $f_{X_1|Z}(x_1|z)$.

$$f_{X_1|Z}(x_1|z) = \frac{f_{ZX_1}(z, x_1)}{f_Z(z)} = \begin{cases} \frac{-1}{x_1\ln(z^2)}, & \text{for } z^2 \le x_1 < 1 \\ 0, & \text{for } 0 < x_1 < z^2 \end{cases}$$

Substituting this result into Equation 20 gives the answer for $E[Y|Z = z]$ via Equation 19.

$$E[Y|Z = z] = \int_{z^2}^{1} x_1 f_{X_1|Z}(x_1|z) dx_1 = \frac{-1}{\ln(z^2)} \int_{z^2}^{1} dx_1 = \frac{z^2 - 1}{\ln(z^2)}$$

In conclusion, we find that the conditional expectation $E[Y|Z = z]$ is given by the following expression.

$$\boxed{E[Y|Z = z] = \frac{z^2 - 1}{\ln(z^2)}, \text{ for all } 0 < z < 1}$$

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