EN 257: Applied Stochastic Processes
Problem Set 5

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Problem 6.37

The members of a sequence of jointly independent random variables $X[n]$ have probability density functions of the following form.

$$f_X(x;n) = \left(1 - \frac{1}{n}\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}\left(x - \frac{n-1}{n}\sigma\right)^2\right] + \frac{1}{n}\sigma \exp(-\sigma x)u(x)$$

Determine whether or not the random sequence $X[n]$ converges in

(a) the mean-square sense,
(b) probability,
(c) distribution.

Part (a)

Recall, from Definition 6.7-5 on page 379 in [4], that a random sequence $X[n]$ converges in the mean-square sense to the random variable X if

$$\lim_{n\to\infty} E\{|X[n] - X|^2\} = 0.$$

Following the derivation on page 420-421 in [2], we note the Cauchy criterion requires that the following condition must hold in order for mean-square convergence.

$$\lim_{n\to\infty} E\{|X[n] - X|^2\} = 0 \iff \lim_{n\to\infty,\ m\to\infty} E\{|X[n] - X[m]|^2\} = 0$$

For the real-valued random sequence $X[n]$, we have the following result.

$$\begin{aligned} E\{|X[n] - X[m]|^2\} &= E\{(X[n] - X[m])^2\} \\ &= E\{X[n]^2\} - 2E\{X[n]X[m]\} + E\{X[m]^2\} \\ &= E\{X[n]^2\} - 2E\{X[n]\}E\{X[m]\} + E\{X[m]^2\}, \text{ for } n \neq m \end{aligned} \quad (1)$$

Note that in the previous expression we have substituted $E\{X[n]X[m]\} = E\{X[n]\}E\{X[m]\}$, since $\{X[n]\}$ are jointly independent and the expression will be nonzero only for the case $n \neq m$. Substituting the integral expressions for the expectations, we find

$$E\{|X[n] - X[m]|^2\} = \int_{-\infty}^{\infty} x^2 f_X(x;n)dx - 2\left[\int_{-\infty}^{\infty} x f_X(x;n)dx\right]\left[\int_{-\infty}^{\infty} x f_X(x;m)dx\right] + \int_{-\infty}^{\infty} x^2 f_X(x;m)dx. \quad (2)$$

At this point we require the following solutions for the integrals in the previous expression.

$$\begin{aligned}\int_{-\infty}^{\infty} x f_X(x;n)dx &= \left(1-\frac{1}{n}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x\exp\left[-\frac{1}{2\sigma^2}\left(x-\frac{n-1}{n}\sigma\right)^2\right]dx + \frac{\sigma}{n}\int_0^{\infty} x\exp(-\sigma x)dx \\ &= \left(1-\frac{1}{n}\right)^2\sigma + \frac{1}{n\sigma} \end{aligned} \tag{3}$$

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 f_X(x;n)dx &= \left(1-\frac{1}{n}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2\exp\left[-\frac{1}{2\sigma^2}\left(x-\frac{n-1}{n}\sigma\right)^2\right]dx + \frac{\sigma}{n}\int_0^{\infty} x^2\exp(-\sigma x)dx \\ &= \left(1-\frac{1}{n}\right)^3\sigma^2 + \left(1-\frac{1}{n}\right)\sigma^2 + \frac{2}{n\sigma^2} \end{aligned} \tag{4}$$

Substituting Equations 2-4 into Equation 1 yields the following result.

$$\lim_{n\to\infty,\ m\to\infty} E\{|X[n]-X[m]|^2\} = 2\sigma^2 \neq 0$$

In conclusion we find that $X[n]$ does not converge in the mean-square sense.

$$\boxed{\lim_{n\to\infty} E\{|X[n]-X|^2\} \neq 0 \Rightarrow X[n] \overset{m.s.}{\nrightarrow} X}$$

Part (b)

Recall, from Definition 6.7-6 on page 379 in [4], that a random sequence $X[n]$ converges in probability to the limiting random variable X if

$$\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0,\ \forall\epsilon > 0.$$

Let's define the following random sequence $Z[n]$ as follows.

$$Z[n] \triangleq X[n] - X$$

In terms of the PDF $F_Z(z;n)$ of the random sequence $Z[n]$, we have

$$\begin{aligned}\lim_{n\to\infty} P[|X[n]-X| > \epsilon] &= \lim_{n\to\infty} P[|Z[n]| > \epsilon] \\ &= \lim_{n\to\infty}\{P[Z[n] > \epsilon] + P[Z[n] < -\epsilon]\} \\ &= \lim_{n\to\infty}\{1 - F_Z(\epsilon;n) + F_Z(-\epsilon;n)\}.\end{aligned}$$

As a result, we find that the following condition must hold if $X[n]$ converges to X in probability.

$$\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0,\ \forall\epsilon > 0 \iff \lim_{n\to\infty}\{F_Z(\epsilon;n) - F_Z(-\epsilon;n)\} = 1,\ \forall\epsilon > 0$$

While we could evaluate this expression directly from the expressions for $f_X(x;n)$ and the limiting (postulated) form for $f_X(x)$, we know from Part (a) that $X[n]$ cannot converge in probability; that is, as we'll show in Part (c), $X[n]$ converges to a Gaussian random variable with mean σ and variance σ^2. As a result, in the limit of large n, $Z[n] = X[n]-X$ will tend to the difference between two Gaussian random variables – which is well-known to have a mean value equal to the difference of the individual means and a variance equal to the sum of the variances [5].

$$\lim_{n\to\infty}\mu_Z[n]=0 \text{ and } \lim_{n\to\infty}\sigma_Z^2[n]=2\sigma^2 \Rightarrow \lim_{n\to\infty}P[|X[n]-X|>\epsilon]\neq 0,\ \forall\epsilon>0$$

In conclusion we find that $X[n]$ does not converge in probability either.

$$\boxed{\lim_{n\to\infty}P[|X[n]-X|>\epsilon]\neq 0,\ \forall\epsilon>0 \Rightarrow X[n]\overset{P}{\nrightarrow}X}$$

Part (c)

Recall, from Definition 6.7-7 on page 381 in [4], that a random sequence $X[n]$ with PDF $F_X(x;n)$ converges in distribution to the random variable X with PDF $F_X(x)$ if

$$\lim_{n\to\infty}F_X(x;n)=F_X(x)$$

for all x at which $F_X(x;n)$ is continuous. Since convergence in distribution is defined by the limiting behavior of the probability distribution function, we must begin by integrating the pdf $f_X(x;n)$ as follows.

$$\begin{aligned}
F_X(x;n) &= \int_{-\infty}^{x} f_X(\xi;n)d\xi \\
&= \left(1-\frac{1}{n}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\exp\left[-\frac{1}{2\sigma^2}\left(\xi-\frac{n-1}{n}\sigma\right)^2\right]d\xi+\frac{1}{n}\int_{-\infty}^{x}\sigma\exp(-\sigma\xi)u(\xi)d\xi \\
&= \frac{1}{2}\left(1-\frac{1}{n}\right)\left\{1+\operatorname{erf}\left[\frac{x-\left(\frac{n-1}{n}\sigma\right)}{\sqrt{2}\sigma}\right]\right\}+\frac{1}{n}\left(1-e^{-\sigma x}\right)u(x)
\end{aligned}$$

Note that in the previous expression we have used the following well-known integral for a Gaussian density function [5].

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\exp\left[-\frac{1}{2\sigma^2}(\xi-\mu)^2\right]d\xi=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

At this point we can evaluate the limiting behavior of $F_X(x;n)$ for large n.

$$\lim_{n\to\infty}F_X(x;n)=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n}\right)\left\{1+\operatorname{erf}\left[\frac{x-\left(\frac{n-1}{n}\sigma\right)}{\sqrt{2}\sigma}\right]\right\}+\lim_{n\to\infty}\frac{1}{n}\left(1-e^{-\sigma x}\right)u(x)$$

Note that terms with coefficients of $1/n$ tend to zero as n approaches infinity. As a result, we have

$$\lim_{n\to\infty}F_X(x;n)=\lim_{n\to\infty}\frac{1}{2}\left\{1+\operatorname{erf}\left[\frac{x-\left(\frac{n-1}{n}\sigma\right)}{\sqrt{2}\sigma}\right]\right\}=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\sigma}{\sqrt{2}\sigma}\right)\right].$$

In conclusion we find that $X[n]$ converges in distribution such that the following condition holds.

$$\boxed{\lim_{n\to\infty}F_X(x;n)=F_X(x)=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\sigma}{\sqrt{2}\sigma}\right)\right]\Rightarrow X[n]\overset{D}{\rightarrow}X}$$

Problem 6.40

Let $X[n]$ be a real-valued random sequence on $n \geq 0$ composed of stationary and independent increments such that $X[n] - X[n-1] = W[n]$ (i.e., where the increment $W[n]$ is a stationary and independent random sequence). Assume that $X[0] = 0$, $E\{X[1]\} = \eta$, and $\mathrm{Var}\{X[1]\} = \sigma^2$.

(a) Find $\mu_X[n]$ and $\sigma_X^2[n]$ for any time $n > 1$.

(b) Prove that $X[n]/n$ converges in probability to η as the time n approaches infinity.

Part (a)

Following the approach in Problem 6.22, let's begin by determining the general form for $X[n]$. We can evaluate the first few terms in the sequence directly.

$$\begin{aligned} X[1] &= X[0] + W[1] \\ X[2] &= X[0] + W[1] + W[2] \\ X[3] &= X[0] + W[1] + W[2] + W[3] \end{aligned}$$

By inspection, we conclude that the general form for $X[n]$ is given by

$$X[n] = X[0] + \sum_{m=1}^{n} W[m],$$

where $X[0]$ is the homogeneous solution to $X[n] = X[n-1]$. Substituting the initial condition $X[0] = 0$ yields the specific solution for $X[n]$.

$$X[n] = \sum_{m=1}^{n} W[m] \tag{5}$$

At this point we recall, from page 319 in [4], that the mean function of a random sequence is given by the following expression.

$$\mu_X[n] \triangleq E\{X[n]\}$$

Substituting Equation 5 and exploiting the linearity of the expectation operator, we find

$$\begin{aligned} \mu_X[n] &= E\left\{\sum_{m=1}^{n} W[m]\right\} = \sum_{m=1}^{n} E\{W[m]\} = \sum_{m=1}^{n} E\{W[1]\} \\ &= \sum_{m=1}^{n} E\{X[1] - X[0]\} = \sum_{m=1}^{n} E\{X[1]\} = \sum_{m=1}^{n} \eta = n\eta. \end{aligned} \tag{6}$$

Note that in the previous expression we have applied the condition that $W[n]$ is a stationary process to conclude that $E\{W[n]\} = E\{W[1]\}$ (since, by Theorem 6.1-2, all stationary random sequences are also wide-sense stationary and, by Definition 6.1-6, all wide-sense stationary processes have a constant mean function [4]). Similarly, we recall that the variance function is given by the following expression.

$$\sigma_X^2[n] = \mathrm{Var}\{X[n]\} \triangleq E\{(X[n] - \mu_X[n])(X[n] - \mu_X[n])^*\}$$

Substituting our previous results and assuming $X[n]$ is a real-valued sequence, we find

$$\begin{aligned}
\sigma_X^2[n] &= E\{(X[n]-\mu_X[n])^2\} \\
&= E\left\{\left[\left(\sum_{m=1}^{n} W[m]\right) - n\eta\right]^2\right\} \\
&= E\left\{\left[\left(\sum_{l=1}^{n} W[l]\right) - n\eta\right]\left[\left(\sum_{m=1}^{n} W[m]\right) - n\eta\right]\right\} \\
&= \sum_{l=1}^{n}\sum_{m=1}^{n} E\{W[l]W[m]\} - 2n\eta\sum_{m=1}^{n} E\{W[m]\} + n^2\eta^2.
\end{aligned}$$

Note that in the previous expression we have exploited the linearity property of the expectation operator. At this point we recall that $W[n]$ is a stationary independent sequence, such that $E\{W[n]\} = E\{W[1]\} = \eta$, and must satisfy the following condition.

$$E\{W[l]W[m]\} = \begin{cases} E\{W[1]^2\}, & \text{for } l = m \\ E\{W[1]\}E\{W[1]\} = \eta^2, & \text{otherwise} \end{cases}$$

Substituting this condition into the previous expression yields the following result.

$$\begin{aligned}
\sigma_X^2[n] &= \sum_{m=1}^{n} E\{W[m]^2\} + (n^2-n)\eta^2 - 2n^2\eta^2 + n^2\eta^2 \\
&= \sum_{m=1}^{n} E\{(W[m]-\eta)^2\} = \sum_{m=1}^{n} \mathrm{Var}\{W[m]\} = \sum_{m=1}^{n} \mathrm{Var}\{W[1]\} \\
&= n\mathrm{Var}\{W[1]\} = nE\{(W[1]-\eta)^2\} = nE\{(X[1]-X[0]-\eta)^2\} \\
&= nE\{(X[1]-E\{X[1]\})^2\} = n\mathrm{Var}\{X[1]\} = n\sigma^2 \qquad (7)
\end{aligned}$$

In conclusion, we find that the mean and variance functions for the random sequence $X[n]$ are given by Equations 6 and 7, respectively.

$$\boxed{\begin{aligned} \mu_X[n] &= n\eta, \text{ for } n > 1 \\ \sigma_X^2[n] &= n\sigma^2, \text{ for } n > 1 \end{aligned}}$$

Part (b)

Recall, from Definition 6.7-6 on page 379 in [4], that a random sequence $X[n]$ converges in probability to the limiting random variable X if

$$\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0, \ \forall\epsilon > 0. \qquad (8)$$

Furthermore, by Chebyshev's inequality, we recall that mean-square convergence such that

$$\lim_{n\to\infty} E\{|X[n]-X|^2\} = 0$$

implies convergence in probability, since

$$P[|X[n]-X| > \epsilon] \le E\{|X[n]-X|^2\}/\epsilon^2, \ \forall\epsilon > 0.$$

As a result, we proceed by proving that the real-valued sequence $X[n]/n$ converges in the mean-square sense to the constant $X = \eta$.

$$\begin{aligned}
E\left\{\left|\frac{X[n]}{n}-\eta\right|^2\right\} &= E\left\{\left(\frac{X[n]}{n}-\eta\right)^2\right\} \\
&= \frac{1}{n^2}E\left\{X[n]^2\right\} - \frac{2\eta}{n}E\left\{X[n]\right\} + \eta^2 \\
&= \frac{1}{n^2}E\left\{X[n]^2\right\} - \eta^2 = \frac{1}{n^2}E\left\{(X[n]-n\eta)^2\right\} \\
&= \frac{1}{n^2}E\left\{(X[n]-E\{X[n]\})^2\right\} = \frac{1}{n^2}\sigma_X^2[n] = \frac{\sigma^2}{n}
\end{aligned}$$

Substituting into Equation 8, we find that $X[n]/n$ converges in the mean-square sense to $X = \eta$.

$$\lim_{n\to\infty} E\left\{\left|\frac{X[n]}{n}-\eta\right|^2\right\} = 0 \Rightarrow \frac{X[n]}{n} \stackrel{m.s.}{\rightarrow} \eta$$

In conclusion, since mean-square converge implies converge in probability, we conclude that $X[n]/n$ converges in probability to η as the time n approaches infinity.

$$\boxed{\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0,\ \forall \epsilon > 0 \Rightarrow \frac{X[n]}{n} \stackrel{P}{\rightarrow} \eta}$$

(QED)

Problem 5.4-9 [Larson and Shubert, p. 341]

Consider a population of mice in some fixed geographical area and let $X[n]$ be their number at the beginning of the n^{th} time period. Assume that during each time period each mouse present at the beginning of that period has a fixed probability p of dying, independently of all the others. Before the end of the n^{th} period, however, a random number $Y[n]$ of new mice invades the area, where $Y[n]$ is a Poisson random variable with parameter λ and is independent of $X[n]$.

(a) Find the conditional expectation $E\{X[n+1]|X[n]=x\}$ for all $n \geq 1$.

(b) Use the conditional expectation to obtain a recurrence relation for $\mu_X[n] = E\{X[n]\}$.

(c) Show the average number of mice $\mu_X[n]$ approaches a limit as $n \to \infty$ and evaluate this limit.

Part (a)

Let's begin by defining the number of mice $X[n+1]$ at the beginning of time period $n+1$. From the problem statement we have

$$X[n+1] = X[n] - D[n] + Y[n],$$

where $X[n]$ is the number of mice at the beginning of period n, $D[n]$ is the number of mice which died during the previous period, and $Y[n]$ is the number of invading mice. Since the conditional expectation operator is linear, we conclude

$$\begin{aligned} E\{X[n+1]|X[n]=x\} &= E\{X[n]|X[n]=x\} - E\{D[n]|X[n]=x\} + E\{Y[n]|X[n]=x\} \\ &= x - E\{D[n]|X[n]=x\} + E\{Y[n]|X[n]=x\}. \end{aligned} \tag{9}$$

To proceed we must determine the remaining conditional expectations in Equation 9. First, note that $D[n]$, the number of mice that died in period n, follows a binomial distribution.

$$P\{D[n]=d|X[n]=x\} = \binom{x}{d} p^d (1-p)^{x-d}$$

As a result, we conclude that the expected number of deaths is given by the following expression in $X[n]=x$ and p.

$$E\{D[n]|X[n]=x\} = \sum_{d=0}^{x} d \binom{x}{d} p^d (1-p)^{x-d} = px \tag{10}$$

Similarly, from the problem statement, we note that $Y[n]$ follows a Poisson distribution with parameter λ.

$$P\{Y[n]=y|X[n]=x\} = \frac{\lambda^y e^{-\lambda}}{y!}$$

Following Example 4.1-2 on page 173 in [4], we conclude that the expected number of invading mice is given by the following expression in $X[n]=x$ and λ.

$$E\{Y[n]|X[n]=x\} = \sum_{y=0}^{\infty} y \left(\frac{\lambda^y e^{-\lambda}}{y!}\right) = \lambda \tag{11}$$

Substituting Equations 10 and 11 into Equation 9 yields the desired expression for the conditional expectation.

$$\boxed{E\{X[n+1]|X[n]=x\} = (1-p)x + \lambda} \tag{12}$$

Part (b)

Recall from Problem 6.22 that the mean function $\mu_X[n]$ is given by

$$\mu_X[n] = E\{X[n]\}.$$

For the initial condition $X[1]$ we must have

$$\mu_X[1] = E\{X[1]\} = X[1],$$

since $X[1]$ is a known constant. By recursively applying the conditional expectation in Equation 12, we can determine the first few terms of $\mu_X[n]$.

$$\begin{aligned}\mu_X[2] &= E\{X[2]|X[1]\} = (1-p)X[1] + \lambda \\ \mu_X[3] &= E\{X[3]|X[2]\} = (1-p)^2 X[1] + (1-p)\lambda + \lambda\end{aligned}$$

By induction, we conclude that $\mu_X[n]$ is given by the following expression.

$$\mu_X[n] = \begin{cases} (1-p)^{n-1}X[1] + \lambda \sum_{i=0}^{n-2}(1-p)^i, & \text{for } n > 1 \\ X[1], & \text{for } n = 1 \end{cases}$$

For $0 < p \leq 1$ the geometric series converges and $\mu_X[n]$ has the following solution.

$$\boxed{\mu_X[n] = \begin{cases} \frac{\lambda}{p} + (1-p)^{n-1}\left(X[1] - \frac{\lambda}{p}\right), & \text{for } n > 1 \\ X[1], & \text{for } n = 1 \end{cases}} \tag{13}$$

Part (c)

For $0 < p \leq 1$ the average number of mice $\mu_X[n]$ approaches a finite limit as $n \to \infty$.

$$\lim_{n\to\infty} \mu_X[n] = \lim_{n\to\infty} \left\{\frac{\lambda}{p} + (1-p)^{n-1}\left(X[1] - \frac{\lambda}{p}\right)\right\} = \frac{\lambda}{p}$$

Note that, since $0 < p \leq 1$, then $(1-p)^{n-1}$ tends to zero as n becomes large. As a result we conclude that, regardless of the starting population $X[1]$, the average number of mice $\mu_X[n]$ approaches the following limit as $n \to \infty$.

$$\boxed{\lim_{n\to\infty} \mu_X[n] = \frac{\lambda}{p}, \text{ for } 0 < p \leq 1}$$

Obviously, for $p = 0$, there will be no deaths and the population will grow without bound for $\lambda > 0$.

Problem 5.4-13 [Larson and Shubert, p. 342]

Let X_1 and X_2 be independent random variables both uniformly distributed on $(0,1)$. For

$$Y = \frac{1}{2}(X_1 + X_2) \quad \text{and} \quad Z = \sqrt{X_1 X_2}$$

find the conditional expectation $E[Y|Z = z]$ for all $0 < z < 1$.

We begin our analysis by applying the linearity property of the conditional expectation operator.

$$E[Y|Z = z] = E\left[\frac{1}{2}(X_1 + X_2) \;\Big|\; Z = z\right] = \frac{1}{2}E[X_1|Z = z] + \frac{1}{2}E[X_2|Z = z] = E[X_1|Z = z] \quad (14)$$

Note that, on the right-hand side, we have used the fact that the expression is symmetric in X_1 and X_2, so we are only required to evaluate the single conditional expectation $E[X_1|Z = z]$. Now recall, from Equations 4.2-9 and 4.2-10 on page 186 in [4], that the conditional expectation of X_1 given $Z = z$ is

$$E[X_1|Z = z] \triangleq \int_{-\infty}^{\infty} x_1 f_{X_1|Z}(x_1|z) dx_1, \quad (15)$$

where the conditional probability density function is given by

$$f_{X_1|Z}(x_1|z) = \frac{f_{ZX_1}(z, x_1)}{f_Z(z)}, \text{ for } f_Z(z) \neq 0. \quad (16)$$

At this point all that remains is to determine closed-form expressions for $f_{ZX_1}(z, x_1)$ and $f_Z(z)$; substituting these expressions into Equation 16 will yield the desired solution for $E[Y|Z = z]$ via Equations 14 and 15.

The expression for $f_Z(z)$ can be found using the approach outlined in Example 3.3-1 on page 186. Specifically, we know that the following expression defines the distribution function $F_Z(z)$.

$$F_Z(z) = \int\int_{(x_1,x_2)\,\in\, C_z} f_{X_1X_2}(x_1, x_2) dx_1 dx_2, \text{ for } \{Z \leq z\} = \{(X_1, X_2) \in C_z\}$$

For X_1 and X_2 uniformly distributed on $(0,1)$, the joint density function has the following form.

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1, & \text{for } 0 < x_1, x_2 < 1 \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

To evaluate the previous expression for $F_Z(z)$, we note that X_2 can be expressed in terms of X_1 using $Z = \sqrt{X_1 X_2}$. As a result, we must have

$$x_2 = \begin{cases} \frac{z^2}{x_1}, & \text{for } z^2 \leq x_1 < 1 \\ 1, & \text{for } 0 < x_1 < z^2 \end{cases}$$

which yields the following result for the probability distribution $F_Z(z)$.

$$F_Z(z) = \int_{z^2}^{1} \left(\int_0^{\frac{z^2}{x_1}} dx_2 \right) dx_1 + \int_0^{z^2} \left(\int_0^1 dx_2 \right) dx_1 = \left[1 - \ln(z^2)\right] z^2$$

Taking the first derivative with respect to z yields the desired expression for $f_Z(z)$.

$$f_Z(z) = \frac{dF_Z(z)}{dz} = -2z\ln(z^2) \tag{18}$$

The expression for $f_{ZX_1}(z, x_1)$ can be found using the approach outlined in Example 3.5-4 on page 159. We begin by defining the pair of random variables Z and X_1 as functions of X_1 and X_2.

$$Z \triangleq g(X_1, X_2) = \sqrt{X_1 X_2} \quad \text{and} \quad X_1 \triangleq h(X_1, X_1) = X_1$$

Next, we observe that the equations

$$z - g(x_1, x_2) = 0 \quad \text{and} \quad x_1 - h(x_1, x_2) = 0$$

have only one real root, for $0 < x_1, x_2 < 1$, given by

$$x_1^1 = \phi_1(z, x_1) = x_1 \quad \text{and} \quad x_2^1 = \varphi_1(z, x_1) = \tfrac{z^2}{x_1}. \tag{19}$$

At this point we recall that $f_{ZX_1}(z, x_1)$ can be obtained directly from $f_{X_1X_2}(x_1, x_2)$ using the methods outlined in Section 3.4 in [4]. From that section we note that the joint pdf can be expressed as

$$f_{ZX_1}(z, x_1) = \sum_{i=1}^{n} f_{X_1X_2}(x_1^i, x_2^i)|\tilde{J}_i|, \tag{20}$$

where $|\tilde{J}_i|$ is the magnitude of the Jacobian transformation such that

$$|\tilde{J}_i| = \left| \det \begin{pmatrix} \partial\phi_i/\partial z & \partial\phi_i/\partial x_1 \\ \partial\varphi_i/\partial z & \partial\varphi_i/\partial x_1 \end{pmatrix} \right| \tag{21}$$

and n is the number of solutions to the equations $z = g(x_1, x_2)$ and $x_1 = h(x_1, x_2)$. Substituting Equation 19 into Equation 21 gives the following Jacobian magnitude.

$$|\tilde{J}_1| = \left| \det \begin{pmatrix} 0 & 1 \\ \frac{2z}{x_1} & -\frac{z^2}{x_1^2} \end{pmatrix} \right| = \frac{2z}{x_1} \tag{22}$$

Substituting Equations 17, 19, and 22 into Equation 20 yields the desired expression for the joint probability density $f_{ZX_1}(z, x_1)$.

$$f_{ZX_1}(z, x_1) = \left(\frac{2z}{x_1}\right) f_{X_1X_2}\left(\frac{z^2}{x_1}, x_1\right) = \begin{cases} \frac{2z}{x_1}, & \text{for } z^2 \le x_1 < 1 \\ 0, & \text{for } 0 < x_1 < z^2 \end{cases} \tag{23}$$

Now that we have determined closed-form expressions for $f_{ZX_1}(z, x_1)$ and $f_Z(z)$, we can substitute into Equation 16 to obtain the conditional density $f_{X_1|Z}(x_1|z)$.

$$f_{X_1|Z}(x_1|z) = \frac{f_{ZX_1}(z, x_1)}{f_Z(z)} = \begin{cases} \frac{-1}{x_1 \ln(z^2)}, & \text{for } z^2 \le x_1 < 1 \\ 0, & \text{for } 0 < x_1 < z^2 \end{cases}$$

Substituting this result into Equation 15 gives the answer for $E[Y|Z = z]$ via Equation 14.

$$E[Y|Z = z] = \int_{z^2}^{1} x_1 f_{X_1|Z}(x_1|z)dx_1 = \frac{-1}{\ln(z^2)} \int_{z^2}^{1} dx_1 = \frac{z^2 - 1}{\ln(z^2)}$$

In conclusion, we find that the conditional expectation $E[Y|Z = z]$ is given by the following expression.

$$\boxed{E[Y|Z = z] = \frac{z^2 - 1}{\ln(z^2)}, \text{ for all } 0 < z < 1}$$

Problem 7.1-7 [Larson and Shubert, p. 426]

Let $X[n]$, for $n = 1, 2, \ldots$, be a sequence of independent random variables with $E\{X[n]\} = \mu_X[n]$ and $\text{Var}\{X[n]\} = \sigma_X^2[n]$. Use the Cauchy criterion to show that the sequence of partial sums $S[n] = X[1] + \ldots + X[n]$, for $n = 1, 2, \ldots$, converges in mean-square sense if and only if the infinite series $\sum_{n=1}^{\infty} \mu_X[n]$ and $\sum_{n=1}^{\infty} \sigma_X^2[n]$ converge. Can the independence assumption be weakened?

Recall, from Definition 6.7-5 on page 379 in [4], that a random sequence $S[n]$ converges in the mean-square sense to the random variable S if

$$\lim_{n\to\infty} E\{|S[n] - S|^2\} = 0.$$

Following the derivation on page 420-421 in [2], we note the Cauchy criterion requires that the following condition must hold in order for mean-square convergence.

$$\lim_{n\to\infty} E\{|S[n] - S|^2\} = 0 \iff \lim_{n\to\infty,\ m\to\infty} E\{|S[n] - S[m]|^2\} = 0 \tag{24}$$

For the real-valued random sequence $S[n]$, we have the following result.

$$\begin{aligned} E\{|S[n] - S[m]|^2\} &= E\{(S[n] - S[m])^2\} \\ &= E\{S[n]^2\} - 2E\{S[n]S[m]\} + E\{S[m]^2\}, \text{ for } n \neq m \end{aligned} \tag{25}$$

In the previous expression we only need to consider the case $n \neq m$, since $E\{|S[n] - S[m]|^2\} = 0$ for $n = m$. At this point we can solve for the individual terms on the right-hand side as follows.

$$\begin{aligned} E\{S[n]^2\} &= E\left\{\left(\sum_{i=1}^{n} X[i]\right)\left(\sum_{j=1}^{n} X[j]\right)\right\} = \sum_{i=1}^{n}\sum_{j=1}^{n} E\{X[i]X[j]\} \\ &= \sum_{i=1}^{n} E\{X[i]^2\} + \sum_{i=1}^{n}\sum_{j=1}^{n} E\{X[i]\}E\{X[j]\} - \sum_{i=1}^{n} E\{X[i]\}^2 \\ &= \sum_{i=1}^{n} E\left\{(X[i] - E\{X[i]\})^2\right\} + \left(\sum_{i=1}^{n} E\{X[i]\}\right)\left(\sum_{j=1}^{n} E\{X[j]\}\right) \\ &= \sum_{i=1}^{n} \sigma_X^2[i] + \left(\sum_{i=1}^{n} \mu_X[i]\right)^2 \end{aligned} \tag{26}$$

Similarly, $E\{S[n]S[m]\}$ is given by the following expression for $n \neq m$.

$$\begin{aligned} E\{S[n]S[m]\} &= E\left\{\left(\sum_{i=1}^{n} X[i]\right)\left(\sum_{j=1}^{m} X[j]\right)\right\} = \sum_{i=1}^{n}\sum_{j=1}^{m} E\{X[i]X[j]\} \\ &= \sum_{i=1}^{\min(n,m)} E\{X[i]^2\} + \sum_{i=1}^{n}\sum_{j=1}^{m} E\{X[i]\}E\{X[j]\} - \sum_{i=1}^{\min(n,m)} E\{X[i]\}^2 \\ &= \sum_{i=1}^{\min(n,m)} E\left\{(X[i] - E\{X[i]\})^2\right\} + \left(\sum_{i=1}^{n} E\{X[i]\}\right)\left(\sum_{j=1}^{m} E\{X[j]\}\right) \\ &= \sum_{i=1}^{\min(n,m)} \sigma_X^2[i] + \left(\sum_{i=1}^{n} \mu_X[i]\right)\left(\sum_{i=1}^{m} \mu_X[i]\right) \end{aligned} \tag{27}$$

Substituting Equations 26 and 27 into Equation 25 yields the following result.

$$\begin{aligned}\lim_{\substack{n\to\infty\\m\to\infty}} E\{|S[n]-S[m]|^2\} = & \lim_{\substack{n\to\infty\\m\to\infty}} \left\{\sum_{i=1}^{n}\sigma_X^2[i] + \left(\sum_{i=1}^{n}\mu_X[i]\right)^2\right\} - \\ & \lim_{\substack{n\to\infty\\m\to\infty}} 2\left\{\sum_{i=1}^{\min(n,m)}\sigma_X^2[i] + \left(\sum_{i=1}^{n}\mu_X[i]\right)\left(\sum_{i=1}^{m}\mu_X[i]\right)\right\} + \\ & \lim_{\substack{n\to\infty\\m\to\infty}} \left\{\sum_{i=1}^{m}\sigma_X^2[i] + \left(\sum_{i=1}^{m}\mu_X[i]\right)^2\right\}\end{aligned}$$

Note that this expression will have a limit of zero if and only if the infinite series $\sum_{n=1}^{\infty}\mu_X[n]$ and $\sum_{n=1}^{\infty}\sigma_X^2[n]$ converge. As a result, by the Cauchy criterion given in Equation 24, the random sequence $S[n]$ will converge in the mean-square sense if and only if the infinite series $\sum_{n=1}^{\infty}\mu_X[n]$ and $\sum_{n=1}^{\infty}\sigma_X^2[n]$ converge.

$$\therefore \boxed{\lim_{n\to\infty} E\{|S[n]-S|^2\} = 0 \Rightarrow S[n] \overset{m.s.}{\to} S \text{ iff } \sum_{n=1}^{\infty}\mu_X[n] \text{ and } \sum_{n=1}^{\infty}\sigma_X^2[n] \text{ converge}}$$

To complete our analysis we note that the independence assumption could be lifted in certain cases. Examining Equation 25 we note that the following condition must hold for $S[n]$ to converge in the mean-square sense.

$$\lim_{\substack{n\to\infty\\m\to\infty}} E\{|S[n]-S[m]|^2\} = \lim_{\substack{n\to\infty\\m\to\infty}} \left\{E\{S[n]^2\} - 2E\{S[n]S[m]\} + E\{S[m]^2\}\right\}$$

Since $R_{SS}[n,m] = E\{S[n]S[m]\}$, we conclude that the following general condition must hold in order for $S[n]$ to converge in the mean-square sense. (Note that this result is also known as the Loève criterion [2]).

$$\boxed{\text{(Loève criterion) } S[n] \overset{m.s.}{\to} S \text{ iff } \lim_{\substack{n\to\infty\\m\to\infty}} R_{SS}[n,m] = C, \text{ for } C \in \mathbb{R}}$$

In other words, we find that under general conditions (i.e., even if $\{X[n]\}$ are not independent) the sequence of partial sums $\{S[n]\}$ will converge if the autocovariance function $R_{SS}[n,m]$ approaches a fixed constant C as n and m become large.

Problem 7.1-10 [Larson and Shubert, p. 427]

If $X[n]$, for $n = 1, 2, \ldots$, is a sequence of i.i.d. random variables with zero means and unit variances, it follows from the central limit theorem that the stochastic sequence

$$S[n] = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X[k], \text{ for } n = 1, 2, \ldots,$$

converges in distribution to a standard Gaussian random variable. Does the sequence $S[n]$, for $n = 1, 2, \ldots$, also converge in the mean-square sense? Can it possibly converge almost surely?

As in Problem 7.1-7 we recall, from Definition 6.7-5 on page 379 in [4], that a random sequence $S[n]$ converges in the mean-square sense to the random variable S if

$$\lim_{n\to\infty} E\{|S[n] - S|^2\} = 0.$$

Following the derivation on page 420-421 in [2], we note the Cauchy criterion requires that the following condition must hold in order for mean-square convergence.

$$\lim_{n\to\infty} E\{|S[n] - S|^2\} = 0 \iff \lim_{n\to\infty,\ m\to\infty} E\{|S[n] - S[m]|^2\} = 0$$

For the real-valued random sequence $S[n]$, we have the following result.

$$E\{|S[n] - S[m]|^2\} = E\{S[n]^2\} - 2E\{S[n]S[m]\} + E\{S[m]^2\}, \text{ for } n \neq m$$

Substituting Equations 26 and 27 from Problem 7.1-7, we conclude that this condition can be expressed as follows.

$$\begin{aligned} E\{|S[n] - S[m]|^2\} = &\frac{1}{n}\left\{\sum_{i=1}^{n} \sigma_X^2[i] + \left(\sum_{i=1}^{n} \mu_X[i]\right)^2\right\} - \\ &\frac{2}{\sqrt{nm}}\left\{\sum_{i=1}^{\min(n,m)} \sigma_X^2[i] + \left(\sum_{i=1}^{n} \mu_X[i]\right)\left(\sum_{i=1}^{m} \mu_X[i]\right)\right\} + \\ &\frac{1}{m}\left\{\sum_{i=1}^{m} \sigma_X^2[i] + \left(\sum_{i=1}^{m} \mu_X[i]\right)^2\right\} \end{aligned}$$

For this problem we have $\mu_X[n] = 0$ and $\sigma_X^2[n] = 1$, such that the following condition holds.

$$E\{|S[n] - S[m]|^2\} = 2\left(1 - \frac{\min(n,m)}{\sqrt{nm}}\right)$$

In conclusion, in the limit of large n and m, the previous expression will converge to zero – implying that $S[n]$ does converge in the mean-square sense via Equation 24.

$$\boxed{\lim_{n\to\infty} E\{|S[n] - S|^2\} = 0 \Rightarrow S[n] \overset{m.s.}{\to} S}$$

In addition, via the Strong Law of Large Numbers given on page 387 in [4], we also conclude that the sequence $S[n]$ converges almost surely to a standard Gaussian random variable.

Problem 7.1-11 [Larson and Shubert, p. 427]

A closed-loop control system is trying to reach the state $X = 0$. It operates in such a manner that if at time n its state is $X[n] = x[n]$, then at time $n+1$ it transitions to state $X[n+1] = x[n] - Z[n]$, where the correction $Z[n]$ is a Gaussian random variable with mean $\mu = x[n]$ and standard deviation $\sigma = \gamma|x[n]|$, for $\gamma > 0$. Thus, the corrections are contaminated by noise proportional to the magnitude of the correction. For what values of the constant γ is the system successful in the sense that $X[n]$ converges to $x = 0$ in the mean-square sense? What happens for other values of γ?

Recall, from Definition 6.7-5 on page 379 in [4], that a random sequence $X[n]$ converges in the mean-square sense to the random variable X if

$$\lim_{n\to\infty} E\{|X[n] - X|^2\} = 0.$$

As a result, we need to demonstrate that the following condition holds in order for the closed-loop control system to reach the state $X = 0$, where $X[n]$ is real-valued.

$$\lim_{n\to\infty} E\{X[n]^2\} = 0 \tag{28}$$

Let's begin our analysis by determining the general form for $X[n]$, where $x[0]$ is the known initial condition. We can evaluate the first few terms in the sequence directly.

$$\begin{aligned} X[1] &= x[0] - Z[0] \\ X[2] &= x[0] - Z[0] - Z[1] \\ X[3] &= x[0] - Z[0] - Z[1] - Z[2] \end{aligned}$$

By inspection, we conclude that the general form for $X[n]$ is given by

$$X[n] = x[0] - \sum_{m=0}^{n-1} Z[m],$$

where $x[0]$ is the homogeneous solution to $X[n+1] = X[n]$. At this point we can determine a closed-form expression for $E\{X[n+1]^2\}$ as follows.

$$\begin{aligned} E\{X[n+1]^2\} &= E\{(X[n] - Z[n])^2\} = E\{(Z[n] - X[n])^2\} \\ &= E\left\{(Z[n] - E\{Z[n]\})^2\right\} \\ &= \mathrm{Var}\{Z[n]\} = \sigma^2 = \gamma^2\, |x[n]|^2 = \gamma^2 E\{X[n]^2\} \end{aligned}$$

From this expression we obtain a simple recurrence relation for $E\{X[n]^2\}$ which has the following solution in terms of the initial condition $x[0]$ and the constant γ.

$$E\{X[n]^2\} = \gamma^{2n}|x[0]|^2$$

Substituting this result into Equation 28 yields the following condition such that $X[n]$ converges to zero in the mean-square sense.

$$\boxed{\lim_{n\to\infty} E\{X[n]^2\} = 0 \Rightarrow X[n] \overset{m.s.}{\to} 0, \text{ for } 0 < \gamma < 1}$$

For $\gamma \geq 1$, a non-empty set of sample paths for $X[n]$ will grow without bound as n increases.

References

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[3] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Cambridge University Press, 2005.

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[5] Wikipedia. Normal distribution. `http://en.wikipedia.org/wiki/Normal_distribution`.