EN 257: Applied Stochastic Processes
Problem Set 6

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Problem 6.22

Let $W[n]$ be an independent random sequence with constant mean $\mu_W = 0$ and variance σ_W^2. Define a new random sequence $X[n]$ as follows:

$$\begin{aligned} X[0] &= 0 \\ X[n] &= \rho X[n-1] + W[n] \text{ for } n \geq 1. \end{aligned}$$

(a) Find the mean value of $X[n]$ for $n \geq 0$.

(b) Find the autocovariance of $X[n]$, denoted as $K_{XX}[m,n]$.

(c) For what values of ρ does $K_{XX}[m,n]$ tend to $G[m-n]$, for some finite-valued function G, as m and n become large? (This situation is known as *asymptotic stationarity.*)

Part (a)

Let's begin by determining the general form for $X[n]$. Following the derivation presented in class, we can evaluate the first few terms in the sequence directly.

$$\begin{aligned} X[1] &= \rho X[0] + W[1] \\ X[2] &= \rho(\rho X[0] + W[1]) + W[2] = \rho^2 X[0] + \rho W[1] + W[2] \\ X[3] &= \rho(\rho^2 X[0] + \rho W[1] + W[2]) + W[3] = \rho^3 X[0] + \rho^2 W[1] + \rho W[2] + W[3] \end{aligned}$$

By inspection, we conclude that the general form for $X[n]$ is given by

$$X[n] = \rho^n X[0] + \sum_{m=1}^{n} \rho^{n-m} W[m],$$

where $\rho^n X[0]$ is the homogeneous solution to $X[n] = \rho X[n-1]$. Substituting the initial condition $X[0] = 0$ yields the specific solution for $X[n]$.

$$X[n] = \sum_{m=1}^{n} \rho^{n-m} W[m] \tag{1}$$

At this point we recall, from page 319 in [5], that the mean function of a random sequence is given by the following expression.

$$\mu_X[n] \triangleq E\{X[n]\}$$

Substituting Equation 1 and exploiting the linearity of the expectation operator, we find

$$\mu_X[n] = E\left\{\sum_{m=1}^{n} \rho^{n-m} W[m]\right\} = \sum_{m=1}^{n} \rho^{n-m} E\{W[m]\} = \sum_{m=1}^{n} \rho^{n-m} \mu_W = 0.$$

As a result, we conclude that the random sequence $X[n]$ is mean-zero for all $n \geq 0$.

$$\boxed{\mu_X[n] = \mu_X = 0, \text{ for } n \geq 0}$$

Part (b)

Recall, from Equation 6.1-10, that the autocovariance $K_{XX}[m,n]$ is defined as follows.

$$K_{XX}[m,n] \triangleq E\{(X[m] - \mu_X[m])(X[n] - \mu_X[n])^*\}$$

Substituting Equation 1 and the result $\mu_X = 0$, we obtain the following expression for $K_{XX}[m,n]$.

$$\begin{aligned} K_{XX}[m,n] &= E\left\{\left(\sum_{i=1}^{m} \rho^{m-i} W[i]\right)\left(\sum_{j=1}^{n} \rho^{n-j} W[j]\right)^*\right\} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \rho^{m-i}(\rho^*)^{n-j} E\left\{W[i]W^*[j]\right\} \end{aligned} \tag{2}$$

At this point, we recall that the variance $\sigma_W^2[n]$ of $W[n]$ is given by the following expression.

$$\sigma_W^2[n] = \text{Var}\left\{W[n]\right\} \triangleq E\left\{(W[n] - \mu_W[n])(W[n] - \mu_W[n])^*\right\}$$

Since $\mu_W[n] = 0$, we have

$$\sigma_W^2[n] = \sigma_W^2 = E\left\{W[n]W^*[n]\right\}, \text{ for } n \geq 0.$$

In addition, we recall from Definition 6.1-2 that an independent random sequence is one whose random variables at any times $\{n_1, n_2, \ldots, n_N\}$ are jointly independent for all positive integers N. As a result, we conclude that $E\left\{W[m]W^*[n]\right\}$ is given by the following expression.

$$E\left\{W[m]W^*[n]\right\} = \begin{cases} \sigma_W^2, & \text{for } m = n \\ 0, & \text{otherwise} \end{cases}$$

Substituting this result into Equation 2 gives the following expression for $K_{XX}[m,n]$.

$$K_{XX}[m,n] = \begin{cases} \sum_{i=1}^{n} \rho^{m-i}(\rho^*)^{n-i}\sigma_W^2, & \text{for } m \geq n \\ \sum_{i=1}^{m} \rho^{m-i}(\rho^*)^{n-i}\sigma_W^2, & \text{for } m < n \end{cases}$$

Following the derivation in class, we conclude that these geometric series will converge for $|\rho| < 1$, such that the solution for $K_{XX}[m,n]$ is given by the following expression.

$$\boxed{K_{XX}[m,n] = \begin{cases} \left[\frac{\rho^{m-n}(1-|\rho|^{2n})}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m \geq n \\ \left[\frac{(\rho^*)^{n-m}(1-|\rho|^{2m})}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m < n \end{cases}, \text{ for } |\rho| < 1}$$

As an aside, we note that $|\rho| < 1$ is a reasonable assumption, since this ensures bounded-input/bounded-output (BIBO) stability. Also, for $\rho \in \mathbb{R}$, this solution reduces to that found in class.

Part (c)

Finally, we conclude by noticing that $X[n]$ is asymptotically stationary for $|\rho| < 1$. That is, in the limit that m and n become large, $K_{XX}[m,n]$ is only a function of the time shift $m - n$ such that

$$\lim_{m\to\infty,\ n\to\infty} K_{XX}[m,n] = G[m-n] = \begin{cases} \left[\frac{\rho^{m-n}}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m \geq n \\ \left[\frac{(\rho^*)^{n-m}}{1-|\rho|^2}\right]\sigma_W^2, & \text{for } m < n \end{cases}, \text{ for } |\rho| < 1$$

Problem 6.25

Consider a wide sense stationary random sequence $X[n]$ input to a linear filter with impulse response

$$h[n] = \begin{cases} 1/2, & n = \{0, 1\} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Write the PSD of the output sequence $S_{YY}(\omega)$ in terms of the PSD of the input sequence $S_{XX}(\omega)$.

(a) Show that the PSD is real-valued, even if $X[n]$ is a complex-valued random sequence.

(b) Show that if $X[n]$ is real-valued, then $S_{XX}(\omega) = S_{XX}(-\omega)$.

(c) Show that $S_{XX}(\omega) \geq 0$ for every ω, regardless of whether $X[n]$ is complex-valued or not.

Let's begin by determining the PSD of the output sequence $S_{YY}(\omega)$ in terms of the PSD of the input sequence $S_{XX}(\omega)$. First, we recall that the autocorrelation of the output sequence $R_{YY}[m]$ is given by Equation 6.4-1 on page 350 in [5].

$$R_{YY}[m] = g[m] * R_{XX}[m], \text{ for } g[m] \triangleq h[m] * h^*[-m]$$

Note that $g[m]$, the *autocorrelation impulse response*, is given by

$$g[m] = h[m] * h^*[-m] = \sum_{k=-\infty}^{\infty} h[k]h^*[m-k] = \begin{cases} 1/4, & m = \{0, 2\} \\ 1/2, & m = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Also recall that the power spectral density (PSD) of the input sequence $S_{XX}(\omega)$ is defined as the discrete-time Fourier transform (DFT) of the input autocorrelation function $R_{XX}[m]$.

$$S_{XX}(\omega) \triangleq \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi \tag{4}$$

Similarly, the PSD of the output sequence $S_{YY}(\omega)$ is given by the DFT of $R_{YY}(\omega)$.

$$S_{YY}(\omega) \triangleq \sum_{m=-\infty}^{\infty} R_{YY}[m]e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi$$

Substituting the previous expression for $R_{YY}[m]$, we find

$$S_{YY}(\omega) = \sum_{m=-\infty}^{\infty} (h[k] * h^*[-m]) * R_{XX}[m]e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi.$$

From the derivation of Equation 6.4-2b on page 351, we conclude that the general form of the output PSD is given by the following expression.

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega), \text{ for } -\pi \leq \omega \leq \pi \tag{5}$$

At this point we can evaluate the DFT of the impulse response, as defined in Equation 3.

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = \frac{1}{2}\left(1 - e^{-j\omega}\right)$$

The magnitude of $H(\omega)$ is then given by

$$|H(\omega)|^2 = \frac{1}{4}\left(1 - e^{-j\omega}\right)\left(1 - e^{j\omega}\right) = \cos^2\left(\omega/2\right).$$

Substituting this result in Equation 5 gives the desired expression for the PSD of the output.

$$\boxed{S_{YY}(\omega) = \cos^2\left(\omega/2\right) S_{XX}(\omega), \text{ for } -\pi \leq \omega \leq \pi}$$

Part (a)

To prove that the PSD $S_{XX}(\omega)$ is a real-valued function, we begin by proving that the autocorrelation function is conjugate symmetric, such that $R_{XX}[m] = R^*_{XX}[-m]$. From the definition of the autocorrelation function $R_{XX}[m]$, we have

$$R_{XX}[m] = E\{X[k+m]X^*[k]\} = E\{X[k]X^*[k-m]\} = E^*\{X[k-m]X^*[k]\} = R^*_{XX}[-m]. \quad (6)$$

If $S_{XX}(\omega)$ is real-valued, then it must satisfy the following condition.

$$S_{XX}(\omega) = S^*_{XX}(\omega)$$

Substituting Equation 4, we must show that the following equality holds.

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\omega m} &= \left(\sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\omega m}\right)^* \\
&= \sum_{m=-\infty}^{\infty} R^*_{XX}[m]e^{j\omega m} \\
&= \sum_{m=-\infty}^{\infty} R_{XX}[-m]e^{j\omega m} \\
&= \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\omega m}
\end{aligned}$$

Note that in the previous expression we have applied the result of Equation 6 to conclude that $R^*_{XX}[m] = R_{XX}[-m]$. In addition, we have substituted $-m$ for m since the order of summation can be reversed. In conclusion, since the left-hand and right-hand sides of the previous expression are equal, we find that $S_{XX}(\omega)$ is a real-valued function, even if $X[n]$ is a complex-valued sequence.

$$\therefore \boxed{S_{XX}(\omega) = S^*_{XX}(\omega) \Rightarrow S_{XX}(\omega) \in \mathbb{R}, \text{ for } -\pi \leq \omega \leq \pi}$$

Part (b)

Let's begin by using Equation 4 to express $S_{XX}(-\omega)$.

$$S_{XX}(-\omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{j\omega m} = \sum_{m=-\infty}^{\infty} R_{XX}[-m]e^{-j\omega m}$$

Note that, since the order of summation can be reversed, we have substituted $-m$ for m in the right-hand expression. At this point we can apply the result of Equation 6 to conclude $R_{XX}[-m] =$

$R^*_{XX}[m]$. Now recall that the autocorrelation function $R_{XX}[m]$ will be real-valued if $X[n]$ is real-valued, such that

$$R^*_{XX}[m] = E^*\{X[k+m]X^*[k]\} = E\{X[k+m]X[k]\} = R_{XX}[m], \text{ for } \{X[n]\} \in \mathbb{R}.$$

We conclude that $R_{XX}[-m] = R_{XX}[m]$ if $X[n]$ is real-valued and, substituting into the previous expression, we find that $S_{XX}(\omega)$ is an even function if $X[n]$ is real-valued.

$$\therefore \boxed{S_{XX}(\omega) = S_{XX}(-\omega), \text{ for } \{X[n]\} \in \mathbb{R}}$$

Part (c)

Recall, from page 351 in [5], that the inverse Fourier transform of the PSD $S_{XX}(\omega)$ is equal to the autocorrelation function $R_{XX}[m]$.

$$R_{XX}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\omega)e^{j\omega m}d\omega$$

Applying Equation 5, we find that the output autocorrelation function is given by

$$R_{YY}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 S_{XX}(\omega)e^{j\omega m}d\omega.$$

At this point, consider a narrowband low-pass filter $H(\omega)$, with bandwidth 2ϵ, centered at frequency ω_o, where $|\omega_0| < \pi$, and with unity gain in the passband. Substituting into the previous expression and evaluating at $m = 0$, we find

$$R_{YY}[0] = \frac{1}{2\pi}\int_{\omega_o-\epsilon}^{\omega_o+\epsilon} S_{XX}(\omega)d\omega \simeq \frac{\epsilon}{\pi}S_{XX}(\omega_o), \text{ for } -\pi \le \omega_o \le \pi.$$

At this point we recall that $R_{YY}[0]$ is a non-negative function, since

$$R_{YY}[0] = E\{Y[k]Y^*[k]\} = E\{|Y[k]|^2\} \ge 0.$$

Substituting this result into the previous expression, we find that $S_{XX}(\omega) \ge 0$ for every ω, regardless of whether $X[n]$ is complex-valued or not.

$$\therefore \boxed{S_{XX}(\omega) \ge 0, \text{ for } -\pi \le \omega \le \pi}$$

Problem 6.26

Let the WSS random sequence X have the correlation function

$$R_{XX}[m] = 10e^{-\lambda_1|m|} + 5e^{-\lambda_2|m|}$$

with $\lambda_1 > 0$ and $\lambda_2 > 0$. Find the corresponding PSD $S_{XX}(\omega)$ for $|\omega| \leq \pi$.

Substituting into Equation 4, we find that the PSD $S_{XX}(\omega)$ is given by the following expression.

$$\begin{aligned} S_{XX}(\omega) &= \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\omega m} \\ &= \sum_{m=-\infty}^{\infty} \left(10e^{-\lambda_1|m|} + 5e^{-\lambda_2|m|}\right) e^{-j\omega m} \\ &= \left(10 \sum_{m=-\infty}^{\infty} e^{-\lambda_1|m|}e^{-j\omega m}\right) + \left(5 \sum_{m=-\infty}^{\infty} e^{-\lambda_2|m|}e^{-j\omega m}\right) \end{aligned}$$

As an aside, we note that the first term in the previous expression can be further simplified by separating the summation into several components.

$$\begin{aligned} 10 \sum_{m=-\infty}^{\infty} e^{-\lambda_1|m|}e^{-j\omega m} &= 10 + 10 \sum_{m=-\infty}^{-1} e^{-\lambda_1|m|}e^{-j\omega m} + 10 \sum_{m=1}^{\infty} e^{-\lambda_1|m|}e^{-j\omega m} \\ &= 10 + 10 \sum_{m=1}^{\infty} e^{-\lambda_1|m|}e^{j\omega m} + 10 \sum_{m=1}^{\infty} e^{-\lambda_1|m|}e^{-j\omega m} \\ &= 10 + 10 \sum_{m=1}^{\infty} e^{-\lambda_1|m|} \left(e^{j\omega m} + e^{-j\omega m}\right) \\ &= 10 + 20 \sum_{m=1}^{\infty} e^{-\lambda_1 m} \cos(\omega m) \end{aligned}$$

Substituting this result into the previous expression yields the following solution for the PSD $S_{XX}(\omega)$ for $\omega \leq \pi$.

$$\boxed{S_{XX}(\omega) = 15 + \sum_{m=1}^{\infty} \left(20e^{-\lambda_1 m} + 10e^{-\lambda_2 m}\right) \cos(\omega m), \text{ for } |\omega| \leq \pi}$$

Applying additional trigonometric identities, we find that this expression can be further simplified.

$$S_{XX}(\omega) = \frac{10\sinh(\lambda_1)}{\cosh(\lambda_1) - \cos(\omega)} + \frac{5\sinh(\lambda_2)}{\cosh(\lambda_2) - \cos(\omega)}, \text{ for } |\omega| \leq \pi$$

Problem 6.29

Consider the LTI system show in Figure P6.29 on page 395 in [5]. Let $X[n]$ and $V[n]$ be WSS and mutually uncorrelated with zero-mean and PSD's $S_{XX}(\omega)$ and $S_{VV}(\omega)$, respectively.

(a) Find the PSD of the output $S_{YY}(\omega)$.

(b) Find the cross-power spectral density $S_{XY}(\omega)$ between the input X and output Y.

Part (a)

Recall from Equation 4 that the PSD of the input sequence $S_{XX}(\omega)$ is defined as the discrete-time Fourier transform (DFT) of the input autocorrelation function $R_{XX}[m]$.

$$S_{XX}(\omega) \triangleq \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi$$

Similarly, the PSD of the output sequence $S_{YY}(\omega)$ is given by the DFT of $R_{YY}(\omega)$.

$$S_{YY}(\omega) \triangleq \sum_{m=-\infty}^{\infty} R_{YY}[m]e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi$$

In order to proceed, we must first find an expression for $R_{YY}[m]$ in terms of $R_{XX}[m]$ and $R_{VV}[m]$. Let us define

$$Z[n] \triangleq X[n] + V[n]$$

as the input to the LTI system with impulse response $h[n]$. By the definition of the autocorrelation function, we must have the following condition (see Table 7.5-1 on page 444 in [5]).

$$\begin{aligned}
R_{ZZ}[m] &= E\{Z[k+m]Z^*[k]\} \\
&= E\{(X[k+m] + V[k+m])(X[k] + V[k])^*\} \\
&= E\{X[k+m]X^*[k]\} + E\{X[k+m]V^*[k]\} + E\{V[k+m]X^*[k]\} + E\{V[k+m]V^*[k]\} \\
&= E\{X[k+m]X^*[k]\} + E\{V[k+m]V^*[k]\} \\
&= R_{XX}[m] + R_{VV}[m] \qquad (7)
\end{aligned}$$

Note that, since $X[n]$ and $V[n]$ are mutually uncorrelated, we can conclude that $E\{X[k+m]V^*[k]\} = E\{V[k+m]X^*[k]\} = 0$. Now recall the expression for S_{YY} previously derived in Problem 6.25.

$$S_{YY}(\omega) = \sum_{m=-\infty}^{\infty} (h[k] * h^*[-m]) * R_{ZZ}[m]e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi$$

Substituting Equation 7, we find that

$$S_{YY}(\omega) = \sum_{m=-\infty}^{\infty} (h[k] * h^*[-m]) * R_{XX}[m]e^{-j\omega m} + \sum_{m=-\infty}^{\infty} (h[k] * h^*[-m]) * R_{VV}[m]e^{-j\omega m}.$$

Finally, we recall the following properties of the DFT: (1) the DFT of the convolution of two functions is the product of their discrete-time Fourier transforms, and (2) the DFT of the complex conjugate of a time-reversed function is equal to the complex conjugate of the Fourier transform of the original function [3]. In addition, we remember from Equation 4 that the autocorrelation function and the power spectral density are Fourier transform pairs. Applying these conditions to the previous expression gives the following result for the PSD of the output $S_{YY}(\omega)$.

$$\boxed{S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) + |H(\omega)|^2 S_{VV}(\omega), \text{ for } -\pi \leq \omega \leq \pi}$$

Part (b)

Similar to the previous part, we begin by recalling the definition of the cross-power spectral density $S_{XY}(\omega)$ given on page 352 in [5].

$$S_{XY}(\omega) \triangleq \sum_{m=-\infty}^{\infty} R_{XY}[m] e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi \tag{8}$$

In order to evaluate $S_{XY}(\omega)$, we require a closed-form expression for the cross-correlation function $R_{XY}(\omega)$. Following the derivation on page 349, we find the following result.

$$\begin{aligned}
R_{XY}[m,n] &= E\{X[m]Y^*[n]\} \\
&= \sum_{k=-\infty}^{\infty} h^*[n-k] E\{X[m]Z^*[k]\} \\
&= \sum_{k=-\infty}^{\infty} h^*[n-k] E\{X[m](X^*[k] + V^*[k])\} \\
&= \sum_{k=-\infty}^{\infty} h^*[n-k] E\{X[m]X^*[k]\} + \sum_{k=-\infty}^{\infty} h^*[n-k] E\{X[m]V^*[k]\} \\
&= \sum_{k=-\infty}^{\infty} h^*[n-k] R_{XX}[m-k] \\
&= \sum_{k=-\infty}^{\infty} h^*[-l] R_{XX}[(m-n)-l], \text{ for } l \triangleq k-n
\end{aligned}$$

At this point we note that the cross-correlation $R_{XY}[m,n]$ is shift-invariant. As a result, we can define $R_{XY}[m] \triangleq R_{XY}[m,0]$. Substituting this result into the previous expression yields

$$R_{XY}[m] = \sum_{k=-\infty}^{\infty} h^*[-l] R_{XX}[m-l] = h^*[-m] * R_{XX}[m].$$

Now we can evaluate Equation 8 to find the desired expression for the cross-power spectral density.

$$S_{XY}(\omega) = \sum_{m=-\infty}^{\infty} h^*[-m] * R_{XX}[m] e^{-j\omega m}, \text{ for } -\pi \leq \omega \leq \pi$$

Applying the previous properties of the DFT used in Part (a), we conclude that $S_{XY}(\omega)$ is given by the following expression.

$$\boxed{S_{XY}(\omega) = H^*(\omega) S_{XX}(\omega), \text{ for } -\pi \leq \omega \leq \pi}$$

Problem 6.32

Recall that a Markov random sequence $X[n]$ is specified by its first-order pdf $f_X(x;n)$ and its one-step conditional pdf

$$f_X(x_n|x_{n-1};n,n-1) = f_X(x_n|x_{n-1}).$$

(a) Find the two-step pdf $f_X(x_n|x_{n-2})$ for a Markov random sequence in terms of the above functions. For this problem, take $n \geq 2$ for a sequence starting at $n = 0$.

(b) Find the N-step pdf $f_X(x_n|x_{n-N})$ for an arbitrary positive integer N, where $n \geq N$.

Part (a)

Recall from pages 429-430 in [5] that the Chapman-Kolmogorov equations can be used to compute the conditional density of $X[n_3]$ given $X[n_1]$, for $n_3 > n_2 > n_1$. To begin our analysis, we note that the joint pdf can be written as follows.

$$f_X(x_3|x_1;n_3,n_1) = \int_{-\infty}^{\infty} f_X(x_3|x_2,x_1;n_3,n_2,n_1) f_X(x_2,x_1;n_2,n_1)dx_2$$

Dividing both sides of this expression by $f(x_1;n_1)$ yields the following result.

$$f_X(x_3|x_1;n_3,n_1) = \int_{-\infty}^{\infty} f_X(x_3|x_2,x_1;n_3,n_2,n_1) f_X(x_2|x_1;n_2,n_1)dx_2$$

Finally, by applying the Markov property we arrive at the well-known Chapman-Kolmogorov equation for the transition density $f_X(x_3|x_1)$.

$$f_X(x_3|x_1;n_3,n_1) = \int_{-\infty}^{\infty} f_X(x_3|x_2;n_3,n_2) f_X(x_2|x_1;n_2,n_1)dx_2, \text{ for } n_3 > n_2 > n_1 \tag{9}$$

From this condition we can conclude that the two-step pdf $f_X(x_n|x_{n-2})$ for a Markov random sequence is given by the following expression (using the simplified notation in the original problem statement).

$$\boxed{f_X(x_n|x_{n-2}) = \int_{-\infty}^{\infty} f_X(x_n|x_{n-1}) f_X(x_{n-1}|x_{n-2})dx_{n-1}}$$

Part (b)

The N-step pdf $f_X(x_n|x_{n-N})$, for an arbitrary positive integer N, can be found in a similar manner as the previous problem by repeatedly using conditioning (i.e., the chain rule of probability [5]).

$$\boxed{f_X(x_n|x_{n-N}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_X(x_n|x_{n-1}) f_X(x_{n-1}|x_{n-2}) \ldots f_X(x_{n-N+1}|x_{n-N})dx_{n-1}dx_{n-2}\ldots dx_{n-N+1}}$$

Problem 6.33

Consider a Markov random sequence $X[n]$ on $1 \leq n \leq N$ that is statistically described by its first-order pdf $f_X(x;1)$ and its one-step transition (conditional) pdf $f_X(x_n|x_{n-1};n,n-1)$. By the Markov definition and suppressing time variables, we have

$$f_X(x_n|x_{n-1}) = f_X(x_n|x_{n-1}, x_{n-2}, \ldots, x_1) \text{ for } 2 \leq n \leq N.$$

Show that such a Markov sequence is also Markov in the reverse order, such that

$$f_X(x_n|x_{n+1}) = f_X(x_n|x_{n+1}, x_{n+2}, \ldots, x_N) \text{ for } 1 \leq n \leq N-1$$

and, as a result, one can alternatively describe a Markov random sequence by its one-step backward pdf $f_X(x_{n-1}|x_n;n-1,n)$ and its first-order pdf $f_X(x;N)$.

Recall, from Equation 2.6-52 on page 104 in [5], that the following expression relates the conditional pdfs for two random variables X and Y.

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} \tag{10}$$

Furthermore, we recall that the chain rule of probability and the Markov property can be used to express the joint pdf as follows.

$$f_X(x_1, x_2, \ldots, x_{N-1}, x_N) = f_X(x_N|x_{N-1})f_X(x_{N-1}|x_{N-2})\ldots f_X(x_2|x_1)f_X(x_1)$$

From Equation 10 we observe that the following condition can be used to relate the conditional probability density functions.

$$f_X(x_n|x_{n-1}) = \frac{f_X(x_{n-1}|x_n)f_X(x_n)}{f_X(x_{n-1})}$$

Substituting this result into the previous expression yields the following equation.

$$f_X(x_1, x_2, \ldots, x_{N-1}, x_N) = \left(\frac{f_X(x_{N-1}|x_N)f_X(x_N)}{f_X(x_{N-1})}\right)\left(\frac{f_X(x_{N-2}|x_{N-1})f_X(x_{N-1})}{f_X(x_{N-2})}\right)\cdots\left(\frac{f_X(x_1|x_2)f_X(x_2)}{f_X(x_1)}\right)f_X(x_1)$$

Simplifying this expression yields the following result.

$$f_X(x_1, x_2, \ldots, x_{N-1}, x_N) = f_X(x_1|x_2)f_X(x_2|x_3)\ldots f_X(x_{N-1}|x_N)f_X(x_N)$$

Note that this expression has the general form of a Markov sequence in reverse order. As a result, we conclude that a Markov sequence is also Markov in the reverse order, such that

$$f_X(x_n|x_{n+1}) = f_X(x_n|x_{n+1}, x_{n+2}, \ldots, x_N) \text{ for } 1 \leq n \leq N-1$$

and, as a result, one can alternatively describe a Markov random sequence by its one-step backward pdf $f_X(x_{n-1}|x_n;n-1,n)$ and its first-order pdf $f_X(x;N)$.

Problem 6.37

The members of a sequence of jointly independent random variables $X[n]$, for $n \geq 1$, have probability density functions of the following form.

$$f_X(x;n) = \left(1 - \frac{1}{n}\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}\left(x - \frac{n-1}{n}\sigma\right)^2\right] + \frac{1}{n}\sigma \exp(-\sigma x) u(x)$$

Determine whether or not the random sequence $X[n]$ converges in

(a) the mean-square sense,
(b) probability,
(c) distribution.

Part (a)

Recall, from Definition 6.7-5 on page 379 in [5], that a random sequence $X[n]$ converges in the mean-square sense to the random variable X if

$$\lim_{n\to\infty} E\{|X[n] - X|^2\} = 0.$$

Following the derivation on page 420-421 in [2], we note the Cauchy criterion requires that the following condition must hold in order for mean-square convergence.

$$\lim_{n\to\infty} E\{|X[n] - X|^2\} = 0 \iff \lim_{n\to\infty,\ m\to\infty} E\{|X[n] - X[m]|^2\} = 0$$

For the real-valued random sequence $X[n]$, we have the following result.

$$\begin{aligned} E\{|X[n] - X[m]|^2\} &= E\{(X[n] - X[m])^2\} \\ &= E\{X[n]^2\} - 2E\{X[n]X[m]\} + E\{X[m]^2\} \\ &= E\{X[n]^2\} - 2E\{X[n]\}E\{X[m]\} + E\{X[m]^2\}, \text{ for } n \neq m \end{aligned} \tag{11}$$

Note that in the previous expression we have substituted $E\{X[n]X[m]\} = E\{X[n]\}E\{X[m]\}$, since $\{X[n]\}$ are jointly independent and the expression will be nonzero only for the case $n \neq m$. Substituting the integral expressions for the expectations, we find

$$E\{|X[n] - X[m]|^2\} = \\ \int_{-\infty}^{\infty} x^2 f_X(x;n)dx - 2\left[\int_{-\infty}^{\infty} x f_X(x;n)dx\right]\left[\int_{-\infty}^{\infty} x f_X(x;m)dx\right] + \int_{-\infty}^{\infty} x^2 f_X(x;m)dx. \tag{12}$$

At this point we require the following solutions for the integrals in the previous expression.

$$\begin{aligned} \int_{-\infty}^{\infty} x f_X(x;n)dx &= \left(1 - \frac{1}{n}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2\sigma^2}\left(x - \frac{n-1}{n}\sigma\right)^2\right] dx + \frac{\sigma}{n} \int_0^{\infty} x \exp(-\sigma x) dx \\ &= \left(1 - \frac{1}{n}\right)^2 \sigma + \frac{1}{n\sigma} \end{aligned} \tag{13}$$

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 f_X(x;n)dx &= \left(1-\frac{1}{n}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2 \exp\left[-\frac{1}{2\sigma^2}\left(x-\frac{n-1}{n}\sigma\right)^2\right]dx + \frac{\sigma}{n}\int_0^{\infty} x^2\exp(-\sigma x)dx \\ &= \left(1-\frac{1}{n}\right)^3\sigma^2 + \left(1-\frac{1}{n}\right)\sigma^2 + \frac{2}{n\sigma^2} \end{aligned} \tag{14}$$

Substituting Equations 12-14 into Equation 11 yields the following result.

$$\lim_{n\to\infty,\ m\to\infty} E\{|X[n]-X[m]|^2\} = 2\sigma^2 \neq 0$$

In conclusion we find that $X[n]$ does not converge in the mean-square sense.

$$\boxed{\lim_{n\to\infty} E\{|X[n]-X|^2\} \neq 0 \Rightarrow X[n] \stackrel{m.s.}{\nrightarrow} X}$$

Part (b)

Recall, from Definition 6.7-6 on page 379 in [5], that a random sequence $X[n]$ converges in probability to the limiting random variable X if

$$\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0,\ \forall\epsilon > 0.$$

Let's define the following random sequence $Z[n]$ as follows.

$$Z[n] \triangleq X[n] - X$$

In terms of the PDF $F_Z(z;n)$ of the random sequence $Z[n]$, we have

$$\begin{aligned}\lim_{n\to\infty} P[|X[n]-X| > \epsilon] &= \lim_{n\to\infty} P[|Z[n]| > \epsilon] \\ &= \lim_{n\to\infty}\{P[Z[n] > \epsilon] + P[Z[n] < -\epsilon]\} \\ &= \lim_{n\to\infty}\{1 - F_Z(\epsilon;n) + F_Z(-\epsilon;n)\}.\end{aligned}$$

As a result, we find that the following condition must hold if $X[n]$ converges to X in probability.

$$\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0,\ \forall\epsilon > 0 \iff \lim_{n\to\infty}\{F_Z(\epsilon;n) - F_Z(-\epsilon;n)\} = 1,\ \forall\epsilon > 0$$

While we could evaluate this expression directly from the expressions for $f_X(x;n)$ and the limiting (postulated) form for $f_X(x)$, we know from Part (a) that $X[n]$ cannot converge in probability; that is, as we'll show in Part (c), $X[n]$ converges to a Gaussian random variable with mean σ and variance σ^2. As a result, in the limit of large n, $Z[n] = X[n] - X$ will tend to the difference between two Gaussian random variables – which is well-known to have a mean value equal to the difference of the individual means and a variance equal to the sum of the variances [6].

$$\lim_{n\to\infty} \mu_Z[n] = 0 \text{ and } \lim_{n\to\infty} \sigma_Z^2[n] = 2\sigma^2 \Rightarrow \lim_{n\to\infty} P[|X[n]-X| > \epsilon] \neq 0,\ \forall\epsilon > 0$$

In conclusion we find that $X[n]$ does not converge in probability either.

$$\boxed{\lim_{n\to\infty} P[|X[n]-X| > \epsilon] \neq 0,\ \forall\epsilon > 0 \Rightarrow X[n] \stackrel{P}{\nrightarrow} X}$$

Part (c)

Recall, from Definition 6.7-7 on page 381 in [5], that a random sequence $X[n]$ with PDF $F_X(x;n)$ converges in distribution to the random variable X with PDF $F_X(x)$ if

$$\lim_{n\to\infty} F_X(x;n) = F_X(x)$$

for all x at which $F_X(x;n)$ is continuous. Since convergence in distribution is defined by the limiting behavior of the probability distribution function, we must begin by integrating the pdf $f_X(x;n)$ as follows.

$$\begin{aligned}
F_X(x;n) &= \int_{-\infty}^{x} f_X(\xi;n)d\xi \\
&= \left(1-\frac{1}{n}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\exp\left[-\frac{1}{2\sigma^2}\left(\xi-\frac{n-1}{n}\sigma\right)^2\right]d\xi + \frac{1}{n}\int_{-\infty}^{x}\sigma\exp(-\sigma\xi)u(\xi)d\xi \\
&= \frac{1}{2}\left(1-\frac{1}{n}\right)\left\{1+\operatorname{erf}\left[\frac{x-\left(\frac{n-1}{n}\sigma\right)}{\sqrt{2}\sigma}\right]\right\} + \frac{1}{n}\left(1-e^{-\sigma x}\right)u(x)
\end{aligned}$$

Note that in the previous expression we have used the following well-known integral for a Gaussian density function [6].

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\exp\left[-\frac{1}{2\sigma^2}(\xi-\mu)^2\right]d\xi = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

At this point we can evaluate the limiting behavior of $F_X(x;n)$ for large n.

$$\lim_{n\to\infty} F_X(x;n) = \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n}\right)\left\{1+\operatorname{erf}\left[\frac{x-\left(\frac{n-1}{n}\sigma\right)}{\sqrt{2}\sigma}\right]\right\} + \lim_{n\to\infty}\frac{1}{n}\left(1-e^{-\sigma x}\right)u(x)$$

Note that terms with coefficients of $1/n$ tend to zero as n approaches infinity. As a result, we have

$$\lim_{n\to\infty} F_X(x;n) = \lim_{n\to\infty}\frac{1}{2}\left\{1+\operatorname{erf}\left[\frac{x-\left(\frac{n-1}{n}\sigma\right)}{\sqrt{2}\sigma}\right]\right\} = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\sigma}{\sqrt{2}\sigma}\right)\right].$$

In conclusion we find that $X[n]$ converges in distribution such that the following condition holds.

$$\boxed{\lim_{n\to\infty} F_X(x;n) = F_X(x) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\sigma}{\sqrt{2}\sigma}\right)\right] \Rightarrow X[n] \xrightarrow{D} X}$$

Problem 6.40

Let $X[n]$ be a real-valued random sequence on $n \geq 0$ composed of stationary and independent increments such that $X[n] - X[n-1] = W[n]$ (i.e., where the increment $W[n]$ is a stationary and independent random sequence). Assume that $X[0] = 0$, $E\{X[1]\} = \eta$, and $\text{Var}\{X[1]\} = \sigma^2$.

(a) Find $\mu_X[n]$ and $\sigma_X^2[n]$ for any time $n > 1$.

(b) Prove that $X[n]/n$ converges in probability to η as the time n approaches infinity.

Part (a)

Following the approach in Problem 6.22, let's begin by determining the general form for $X[n]$. We can evaluate the first few terms in the sequence directly.

$$\begin{aligned} X[1] &= X[0] + W[1] \\ X[2] &= X[0] + W[1] + W[2] \\ X[3] &= X[0] + W[1] + W[2] + W[3] \end{aligned}$$

By inspection, we conclude that the general form for $X[n]$ is given by

$$X[n] = X[0] + \sum_{m=1}^{n} W[m],$$

where $X[0]$ is the homogeneous solution to $X[n] = X[n-1]$. Substituting the initial condition $X[0] = 0$ yields the specific solution for $X[n]$.

$$X[n] = \sum_{m=1}^{n} W[m] \tag{15}$$

At this point we recall, from page 319 in [5], that the mean function of a random sequence is given by the following expression.

$$\mu_X[n] \triangleq E\{X[n]\}$$

Substituting Equation 15 and exploiting the linearity of the expectation operator, we find

$$\begin{aligned} \mu_X[n] &= E\left\{\sum_{m=1}^{n} W[m]\right\} = \sum_{m=1}^{n} E\{W[m]\} = \sum_{m=1}^{n} E\{W[1]\} \\ &= \sum_{m=1}^{n} E\{X[1] - X[0]\} = \sum_{m=1}^{n} E\{X[1]\} = \sum_{m=1}^{n} \eta = n\eta. \end{aligned} \tag{16}$$

Note that in the previous expression we have applied the condition that $W[n]$ is a stationary process to conclude that $E\{W[n]\} = E\{W[1]\}$ (since, by Theorem 6.1-2, all stationary random sequences are also wide-sense stationary and, by Definition 6.1-6, all wide-sense stationary processes have a constant mean function [5]). Similarly, we recall that the variance function is given by the following expression.

$$\sigma_X^2[n] = \text{Var}\{X[n]\} \triangleq E\{(X[n] - \mu_X[n])(X[n] - \mu_X[n])^*\}$$

Substituting our previous results and assuming $X[n]$ is a real-valued sequence, we find

$$\begin{aligned}
\sigma_X^2[n] &= E\{(X[n]-\mu_X[n])^2\} \\
&= E\left\{\left[\left(\sum_{m=1}^{n} W[m]\right) - n\eta\right]^2\right\} \\
&= E\left\{\left[\left(\sum_{l=1}^{n} W[l]\right) - n\eta\right]\left[\left(\sum_{m=1}^{n} W[m]\right) - n\eta\right]\right\} \\
&= \sum_{l=1}^{n}\sum_{m=1}^{n} E\{W[l]W[m]\} - 2n\eta\sum_{m=1}^{n} E\{W[m]\} + n^2\eta^2.
\end{aligned}$$

Note that in the previous expression we have exploited the linearity property of the expectation operator. At this point we recall that $W[n]$ is a stationary independent sequence, such that $E\{W[n]\} = E\{W[1]\} = \eta$, and must satisfy the following condition.

$$E\{W[l]W[m]\} = \begin{cases} E\{W[1]^2\}, & \text{for } l = m \\ E\{W[1]\}E\{W[1]\} = \eta^2, & \text{otherwise} \end{cases}$$

Substituting this condition into the previous expression yields the following result.

$$\begin{aligned}
\sigma_X^2[n] &= \sum_{m=1}^{n} E\{W[m]^2\} + (n^2-n)\eta^2 - 2n^2\eta^2 + n^2\eta^2 \\
&= \sum_{m=1}^{n} E\{(W[m]-\eta)^2\} = \sum_{m=1}^{n} \mathrm{Var}\{W[m]\} = \sum_{m=1}^{n} \mathrm{Var}\{W[1]\} \\
&= n\mathrm{Var}\{W[1]\} = nE\{(W[1]-\eta)^2\} = nE\{(X[1]-X[0]-\eta)^2\} \\
&= nE\{(X[1]-E\{X[1]\})^2\} = n\mathrm{Var}\{X[1]\} = n\sigma^2
\end{aligned} \tag{17}$$

In conclusion, we find that the mean and variance functions for the random sequence $X[n]$ are given by Equations 16 and 17, respectively.

$$\boxed{\begin{aligned} \mu_X[n] &= n\eta, \text{ for } n > 1 \\ \sigma_X^2[n] &= n\sigma^2, \text{ for } n > 1 \end{aligned}}$$

Part (b)

Recall, from Definition 6.7-6 on page 379 in [5], that a random sequence $X[n]$ converges in probability to the limiting random variable X if

$$\lim_{n\to\infty} P[|X[n]-X| > \epsilon] = 0, \ \forall \epsilon > 0. \tag{18}$$

Furthermore, by Chebyshev's inequality, we recall that mean-square convergence such that

$$\lim_{n\to\infty} E\{|X[n]-X|^2\} = 0$$

implies convergence in probability, since

$$P[|X[n]-X| > \epsilon] \le E\{|X[n]-X|^2\}/\epsilon^2, \ \forall \epsilon > 0.$$

As a result, we proceed by proving that the real-valued sequence $X[n]/n$ converges in the mean-square sense to the constant $X = \eta$.

$$\begin{aligned} E\left\{\left|\frac{X[n]}{n} - \eta\right|^2\right\} &= E\left\{\left(\frac{X[n]}{n} - \eta\right)^2\right\} \\ &= \frac{1}{n^2}E\left\{X[n]^2\right\} - \frac{2\eta}{n}E\left\{X[n]\right\} + \eta^2 \\ &= \frac{1}{n^2}E\left\{X[n]^2\right\} - \eta^2 = \frac{1}{n^2}E\left\{(X[n] - n\eta)^2\right\} \\ &= \frac{1}{n^2}E\left\{(X[n] - E\{X[n]\})^2\right\} = \frac{1}{n^2}\sigma_X^2[n] = \frac{\sigma^2}{n} \end{aligned}$$

Substituting into Equation 18, we find that $X[n]/n$ converges in the mean-square sense to $X = \eta$.

$$\lim_{n\to\infty} E\left\{\left|\frac{X[n]}{n} - \eta\right|^2\right\} = 0 \Rightarrow \frac{X[n]}{n} \overset{m.s.}{\rightarrow} \eta$$

In conclusion, since mean-square converge implies converge in probability, we conclude that $X[n]/n$ converges in probability to η as the time n approaches infinity.

$$\boxed{\lim_{n\to\infty} P[|X[n] - X| > \epsilon] = 0,\ \forall\epsilon > 0 \Rightarrow \frac{X[n]}{n} \overset{P}{\rightarrow} \eta}$$

(QED)

Problem 7.40

Express the answers to the following questions in terms of probability density functions.

(a) State the definition of an independent-increments random process.

(b) State the definition of a Markov random process.

(c) Prove that any random process that has independent increments also has the Markov property.

Part (a)

Recall, from pages 326 and 410 in [5], that a random process has independent increments when the set of n random variables

$$X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are jointly independent for all $t_1 < t_2 < \ldots < t_n$ and $n \geq 1$. In terms of probability density functions, we note that a random process $X(t)$ with independent increments must satisfy

$$f_X(x_n - x_{n-1} | x_{n-1}, x_{n-2}, \ldots, x_1; t_n, t_{n-1}, \ldots, t_1) = f_X(x_n - x_{n-1}; t_n, t_{n-1}), \tag{19}$$

for all $x_1, x_2, \ldots, x_n$ and integers $n > 0$ where $t_1 < t_2 < \ldots < t_n$.

Part (b)

From page 422 in [5] we recall that a (first-order) continuous-valued Markov process $X(t)$ satisfies

$$f_X(x_n | x_{n-1}, x_{n-2}, \ldots, x_1; t_n, t_{n-1}, \ldots, t_1) = f_X(x_n | x_{n-1}; t_n, t_{n-1}), \tag{20}$$

for all $x_1, x_2, \ldots, x_n$ and integers $n > 0$ where $t_1 < t_2 < \ldots < t_n$. Note that this expression defines the so-called one-step conditional pdf, however a continuous-valued Markov process must also satisfy the following k-step pdf given by

$$f_X(x_{n+k} | x_n, x_{n-1}, \ldots, x_1; t_{n+k}, t_n, t_{n-1}, \ldots, t_1) = f_X(x_{n+k} | x_n; t_{n+k}, t_n),$$

for all $x_1, x_2, \ldots, x_n, x_{n+k}$ and integers $n > 0$ and $k > 0$ where $t_1 < t_2 < \ldots < t_n < t_{n+k}$.

Part (c)

Following the derivation outlined on page 423 in [5], we note that any random process $X(t)$ with independent increments must have a pdf that can be expressed in the following form.

$$\begin{aligned}
f_X(x_n | x_{n-1}, x_{n-2}, \ldots, x_1; t_n, t_{n-1}, \ldots, t_1) &= f_X(x_n - x_{n-1} | x_{n-1}, x_{n-2}, \ldots, x_1; t_n, t_{n-1}, \ldots, t_1) \\
&= f_X(x_n - x_{n-1}; t_n, t_{n-1}) \\
&= f_X(x_n - x_{n-1} | x_{n-1}; t_n, t_{n-1}) \\
&= f_X(x_n | x_{n-1}; t_n, t_{n-1})
\end{aligned}$$

Note that, on the first two lines of this expression, we have applied the definition of independent increments given by Equation 19. Since the last line is identical to the definition of Markov random processes given by Equation 20, we conclude that any random process that has independent increments also has the Markov property.

Problem 7.46

Consider the linear system shown in Figure P7.46 on page 485 in [5], which is excited by two *orthogonal* zero-mean, jointly wide-sense stationary random processes $X(t)$, the signal, and $U(t)$, the noise. Let the input to the system G be

$$Y(t) = h(t) * X(t) + U(t),$$

which models a distorted-signal-in-noise estimation problem. If we pass the received signal $Y(t)$ through the filter G, we obtain an estimate $\hat{X}(t)$. Finally, we define the estimation error $\varepsilon(t)$ such that

$$\varepsilon(t) = \hat{X}(t) - X(t).$$

In the following problems we will evaluate some relevant power and cross-power spectral densities.

(a) Find $S_{YY}(\omega)$.
(b) Find $S_{\hat{X}X}(\omega) = S^*_{X\hat{X}}(\omega)$ in terms of $H(\omega)$, $G(\omega)$, $S_{XX}(\omega)$, and $S_{UU}(\omega)$.
(c) Find $S_{\varepsilon\varepsilon}(\omega)$.
(d) Show that, in order to minimize $S_{\varepsilon\varepsilon}(\omega)$ at frequencies where $S_{XX}(\omega) >> S_{UU}(\omega)$, we should select $G \approx H^{-1}$. Similarly, where $S_{XX}(\omega) << S_{UU}(\omega)$, we should have $G \approx 0$.

Part (a)

To begin our analysis we recall that the power spectral density $S_{XX}(\omega)$, for a continuous-valued wide-sense stationary random process $X(t)$, is given on page 443 in [5].

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

Similar to Part (a) of Problem 6.29 and as given by Equation7.5-14, we recall that a LTI system with frequency response $H(\omega)$ has the following output PSD $S_{YY}(\omega)$ for the input process $X(t)$.

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} h(\tau) * R_{XX}(\tau) * h^*(-\tau) e^{-j\omega\tau} d\tau = |H(\omega)|^2 S_{XX}(\omega)$$

Next, we observe that $h(t) * X(t)$ and $U(t)$ are orthogonal since

$$\begin{aligned} E\left\{[h(t) * X(t_1)]\, Y^*(t_2)\right\} &= E\left\{\int_{-\infty}^{\infty} h(\tau) X(t_1 - \tau) Y^*(t_2) d\tau\right\} \\ &= \int_{-\infty}^{\infty} h(\tau) E\left\{X(t_1 - \tau) Y^*(t_2)\right\} d\tau = 0 \end{aligned}$$

and $E\{X(t_1 - \tau)Y^*(t_2)\} = 0$ for the orthogonal random processes $X(t)$ and $Y(t)$ (see page 437). Finally, we recall from Table 7.5-1 that the PSD of two orthogonal random process $X_1(t)$ and $X_2(t)$ is given by $S_{X_1X_1}(\omega) + S_{X_2X_2}(\omega)$. In conclusion $S_{YY}(\omega)$ is given by the following expression.

$$\boxed{S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) + S_{UU}(\omega)}$$

Part (b)

From the problem statement, we have

$$\hat{X}(t) = g(t) * [h(t) * X(t) + U(t)] = g(t) * h(t) * X(t) + g(t) * U(t). \tag{21}$$

Now we recall that the cross-power spectral density $S_{\hat{X}X}(\omega)$ is given by the following expression.

$$S_{\hat{X}X}(\omega) = \int_{-\infty}^{\infty} R_{\hat{X}X}(\tau) e^{-j\omega\tau} d\tau$$

To proceed we need to obtain a closed-form expression for the cross-correlation $R_{\hat{X}X}(\omega)$. By definition, we have

$$\begin{aligned}
R_{\hat{X}X}(\tau) &= E\left\{\hat{X}(t+\tau)X^*(t)\right\} \\
&= E\left\{[g(t+\tau) * h(t+\tau) * X(t+\tau) + g(t+\tau) * U(t+\tau)]\, X^*(t)\right\} \\
&= E\left\{[g(t+\tau) * h(t+\tau) * X(t+\tau)]\, X^*(t)\right\} + E\left\{[g(t+\tau) * U(t+\tau)]\, X^*(t)\right\} \\
&= g(t+\tau) * h(t+\tau) * E\left\{X(t+\tau)X^*(t)\right\} \\
&= g(\tau) * h(\tau) * R_{XX}(\tau),
\end{aligned} \tag{22}$$

since, by definition, $X(t)$ and $U(t)$ are orthogonal random processes such that $R_{UX}(t_1, t_2) = E\{U(t_1)X^*(t_2)\} = 0$ for all t_1 and t_2. Substituting Equation 22 into Equation 21 yields the following solution for $S_{\hat{X}X}(\omega)$.

$$\boxed{S_{\hat{X}X}(\omega) = H(\omega)G(\omega)S_{XX}(\omega)}$$

Part (c)

From the problem statement and Equation 21, we find

$$\varepsilon(t) = \hat{X}(t) - X(t) = [g(t) * h(t) - \delta(t)] * X(t) + g(t) * U(t).$$

Correspondingly, the power spectral density of the estimation error is given by

$$S_{\varepsilon\varepsilon}(\omega) = \int_{-\infty}^{\infty} R_{\varepsilon\varepsilon}(\tau) e^{-j\omega\tau} d\tau.$$

Since the Fourier transform of the Dirac delta function $\delta(t)$ is equal to unity, we conclude that the power spectral density of the estimation error has the following solution.

$$\boxed{S_{\varepsilon\varepsilon}(\omega) = |G(\omega)H(\omega) - 1|^2 S_{XX}(\omega) + |G(\omega)|^2 S_{UU}(\omega)} \tag{23}$$

Part (d)

From Equation 23 we find that, in order to minimize $S_{\varepsilon\varepsilon}(\omega)$ for $S_{XX}(\omega) >> S_{UU}(\omega)$, we must select $G(\omega) \approx [H(\omega)]^{-1}$. Similarly, $G \approx 0$ minimizes $S_{\varepsilon\varepsilon}(\omega)$ for $S_{XX}(\omega) << S_{UU}(\omega)$. In summary, the following conditions on $G(\omega)$ will minimize the power spectral density of the estimation error.

$$\boxed{\begin{gathered}
S_{\varepsilon\varepsilon}(\omega) = \left|[H(\omega)]^{-1} H(\omega) - 1\right|^2 S_{XX}(\omega) + |H(\omega)|^{-2} S_{UU}(\omega) \approx 0, \text{ for } S_{XX}(\omega) >> S_{UU}(\omega) \\
S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega) \approx 0, \text{ for } S_{XX} << S_{UU}(\omega)
\end{gathered}}$$

Problem 7.47

Let $X(t)$, the input to the system in Figure P7.47 on page 486 in [5], be a stationary Gaussian random process. The power spectral density of $Z(t)$ is measured experimentally and found to be

$$S_{ZZ}(\omega) = \pi\delta(\omega) + \frac{2\beta}{(\omega^2+\beta^2)(\omega^2+1)}.$$

(a) Find the correlation function $S_{YY}(\omega)$ in terms of β.

(b) Find the correlation function $S_{XX}(\omega)$.

Part (a)

To begin our analysis we recall that the power spectral density $S_{ZZ}(\omega)$ is given on page 443 in [5].

$$S_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} h(\tau) * R_{YY}(\tau) * h^*(-\tau)e^{-j\omega\tau}d\tau = |H(\omega)|^2 S_{YY}(\omega)$$

To proceed we need to determine a closed-form expression for the system frequency response $H(\omega)$. By definition the frequency response $H(\omega)$ is the discrete-time Fourier transform of the impulse response $h(t)$. As a result, we have

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(1+j\omega)\tau}d\tau = \frac{1}{1+j\omega}.$$

Evaluating the frequency response magnitude $|H(\omega)|^2$, we find

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{1}{\omega^2+1}.$$

Substituting into the previous expression for $S_{ZZ}(\omega)$ yields a solution for $S_{YY}(\omega)$ in terms of β.

$$\boxed{S_{YY}(\omega) = \frac{S_{ZZ}(\omega)}{|H(\omega)|^2}, \text{ for } |H(\omega)| \neq 0 \quad \Rightarrow \quad S_{YY}(\omega) = \pi(\omega^2+1)\delta(\omega) + \frac{2\beta}{\omega^2+\beta^2}}$$

Part (b)

Recall from Table 7.5-1 in [5] that the power spectral density of a random process $X^n(t)$, generated from the random process $X(t)$, is given by $\omega^{2n}S_{XX}(\omega)$. As a result, we find that the PSD of $X^2(t)$ should be given by $\omega^4 S_{XX}(\omega)$. Since $Y(t) = X^2(t)$, we conclude that the correlation function $S_{XX}(\omega)$ has the following form.

$$\boxed{S_{XX}(\omega) = \frac{S_{YY}(\omega)}{\omega^4}, \text{ for } \omega \neq 0 \quad \Rightarrow \quad S_{YY}(\omega) = \pi\left(\frac{\omega^2+1}{\omega^4}\right)\delta(\omega) + \frac{2\beta}{\omega^4(\omega^2+\beta^2)}, \text{ for } \omega \neq 0}$$

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[6] Wikipedia. Normal distribution. `http://en.wikipedia.org/wiki/Normal_distribution`.