Rotations

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There are many ways of representing rotations in 3D. In these notes we will discuss two ways: as 3×3 matrices, and as 3-vectors. An alternative representation is *quaternions*, but we will not study them here.

1 ROTATION MATRICES

A 3D rotation can be represented as a 3×3 matrix

$$R = \begin{pmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{pmatrix}$$

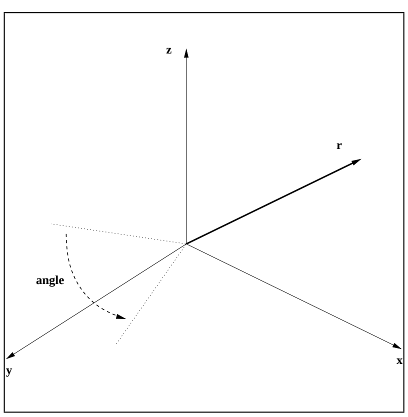
which is *orthogonal* $RR^t = I$ [1] and has unit determinant $|R| = 1$. In particular, the inverse rotation is represented by the transposed matrix $R^{-1} = R^t$. The result of applying the rotation represented by the matrix R to a vector

$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

is computed by multiplying the matrix by the vector Rv. The set of all the 3D rotation matrices, which is denoted $\mathrm{SO}(3)$, is closed under matrix multiplication.

2 ROTATION VECTORS

Another way of representing a 3D rotation is to specify an *axis of rotation* with a unit length vector r, and an *angle of rotation* α in radians. Usually, these two values are jointly specified as a single vector $\omega = \alpha r$. The angle of rotation is the length of ω, and the axis of rotation is determined by normalization $r = \omega/|\omega|$ [2]. Although this is a more intuitive way of representing a 3D rotation, to apply this rotation to a vector we need to convert to the matrix representation.



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[1] I is the 3×3 identity matrix.

[2] We use $|.|$ to denote both determinant of a matrix, and length of a vector.

3 RODRIGUES FORMULA

The conversion from vector to matrix representation is given by *Rodrigues formula*

$$R = c\,I + (1-c)\,rr^t + s\,r^\Lambda \;, \tag{1}$$

where I is the 3×3 identity matrix, $c = \cos(\alpha)$, $s = \sin(\alpha)$, and

$$r^\Lambda = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix} \;.$$

This last matrix is *skew-symmetric* $(r^\Lambda)^t = -r^\Lambda$, and represents the *vector product*: when we multiply the matrix r^Λ by a vector v we obtain the vector product $r \times v$

$$r^\Lambda\, v = r \times v = \begin{pmatrix} r_y\, v_z - r_z\, v_y \\ r_z\, v_x - r_x\, v_z \\ r_x\, v_y - r_y\, v_x \end{pmatrix}$$

Exercice 1 Prove Rodrigues' formula: for every unit-length vector r and angle α, the matrix of equation 1 is a rotation matrix that corresponds to a rotation of α radians around the vector r. (hint: first consider the case $r = (0,0,1)^t$; for the general case consider two other unit-length vectors u and v that jointly with r define an orthogonal matrix $Q = [uvr]$, and study the matrix QRQ^t).

Exercice 2 Implement Rodrigues' formula.

4 INVERSE RODRIGUES FORMULA

To convert from matrix to vector representation we first note that, since $c\,I + (1-c)\,rr^t$ is symmetric, and $s\,r^\Lambda$ skew-symmetric, the transpose of the matrix R in equation 1 is

$$R^t = c\,I + (1-c)\,rr^t - s\,r^\Lambda \;,$$

and so, the skew-symmetric part of R is[3]

$$(R - R^t)/2 = s\,r^\Lambda = (s\,r)^\Lambda \;. \tag{2}$$

The value of s is obtained as the length of the vector $s\,r$. The value of c is obtained by computing a square root $c = \sqrt{1-s^2}$ (why is $s \le 1$?). With c and s we can determine α. And the unit length vector r is obtained by normalization (what happens when $R - R^t = 0$?).

Exercice 3 Implement the inverse of Rodrigues' formula.

[3] Every matrix A can be decomposed as the sum of a *symmetric part*, the matrix $(A + A^t)/2$, and a *skew-symmetric part*, the matrix $(A - A^t)/2$.

5 CONTINUOUS PARAMETERIZATION

Representing a rotation as a vector ω with length equal to the angle of rotation in radians produces multiple representations for each rotation, and the conversion to matrix form requires computation of trigonometric functions. It is better to represent the rotation as a vector with length equal to the positive *sine* of the angle of rotation. Rodrigues' formula defines a function

$$\{\omega : 0 < |\omega| \leq 1\} \rightarrow \mathrm{SO}(3) \; ,$$

with $s = (|\omega|^2)^{1/2}$, $c = (1 - |\omega|^2)^{1/2}$, and $r = \omega/s$, which is clearly $1 - 1$ and continuous.

Exercice 4 Show that the function is well defined and continuous at $\omega = 0$.

Exercice 5 What rotations matrices correspond to $|\omega| = 1$?. What angle of rotation these matrices correspond to ?

The inverse Rodrigues' formula of equation 2 defines the inverse function

$$\omega = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} R_{zx} - R_{xz} \\ R_{xz} - R_{zx} \\ R_{yx} - R_{xy} \end{pmatrix}$$