Fast Non-Convex Hull Computation

Julián Bayardo Spadafora [1]   Francisco Gómez-Fernandez [1]
Gabriel Taubin [2]

[1]Universidad de Buenos Aires

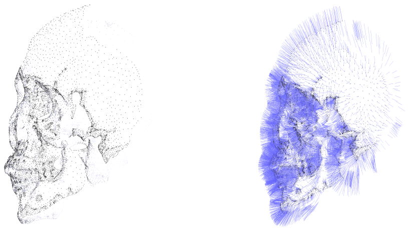
[2]Brown University

International Conference on 3D Vision, 2019

Who am I?

Problem Statement

- 3D surface reconstruction from an *unstructured* point cloud $\mathcal{P} = \{\mathbf{p_1}, \ldots, \mathbf{p_N}\}$ with corresponding *inwards-pointing normal vectors* $\mathcal{N} = \{\mathbf{n_1}, \ldots, \mathbf{n_N}\}, ||\mathbf{n_i}||_2 = 1$
- The sampled surface $\mathcal{S}$ is assumed to be:
  - Watertight (i.e. bounded and closed)
  - Smooth
  - Orientable



Medial Axis Transform (MAT, Blum [2])

- Surface representation as a union of balls
- Skeleton
- Points with more than one shortest path to a surface
- **Maximally contained balls with respect to inclusion**

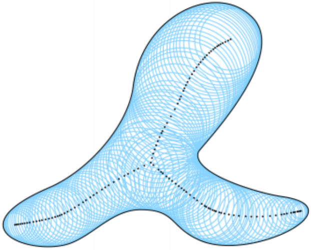


Figure: Example surface and its medial axis transform. By Zhu, Sun, Choi, *et al.* [1]

Non-Convex Hull (NCH, Taubin [3])

- Surface representation as the intersection between complements of balls and linear half-spaces
- Dependant on the surface normal at a given point
  - Equivalent to the MAT (Inner NCH) when normals point inwards
  - Different object (Outer NCH) when normals point outwards

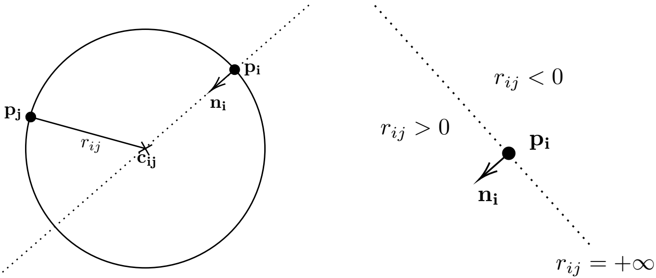
.

Geometric Intuition: Tangent Balls

If $\mathbf{p_i}$ and $\mathbf{p_j}$ are two points in the cloud, with corresponding normals. Then:

- The maximal ball $B_{ij} = \mathcal{B}_{r_{ij}}(\mathbf{c_{ij}})$ tangent to both points, with matching normal $\mathbf{n_i}$ at $\mathbf{p_i}$, is uniquely determined:

$$\mathbf{c_{ij}} = \mathbf{p_i} + \mathbf{n_i} r_{ij} \qquad\qquad r_{ij} = \frac{||\mathbf{p_j} - \mathbf{p_i}||_2^2}{2\langle \mathbf{n_i}, \mathbf{p_j} - \mathbf{p_i}\rangle}$$



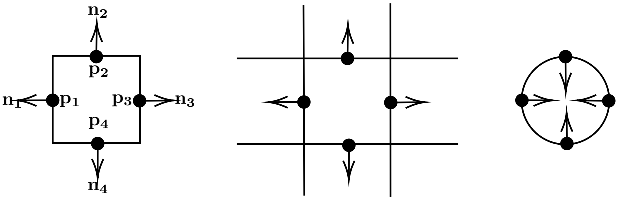
Geometric Intuition: Maximal Balls

- $r_i = \min_{1 \leq j \leq N, i \neq j} \max\{+\infty, r_{ij}\}$ is the *minimum of the maximal radiuses*, with negative values turned into $+\infty$
- B_i is tangent to $\mathbf{p_i}$ and its normal at $\mathbf{p_i}$ matches $\mathbf{n_i}$
- $r_{ik} \neq +\infty$ sets the minimum iff:
  - B_i is tangent to $\mathbf{p_k}$
  - Other points are either outside or tangent to B_i
- $r_i = +\infty$ iff $\mathbf{p_j} \neq \mathbf{p_i}$ either lie on the plane or the other side of the normal

Geometric Intuition: NCH Fits

Two different NCH fits:

- Normals pointing *inwards*, or Inner NCH
  - Any sufficiently dense sample will always find $r_i > 0$
  - Equivalent to the MAT
- Normals pointing *outwards* ($-\mathbf{n_i}$), or Outer NCH
  - Points in flat areas and sharp features will have $r_i = +\infty$
  - $r_i = +\infty$ means that the ball found is a plane



NCH Surface Representation

- Need to combine information from all points together
- One approach is to give a Signed Distance Function $f : \mathbb{R}^3 \to \mathbb{R}$ defined such that:

$$f(\mathbf{x}) = \begin{cases} d(\mathbf{x}, \partial\mathcal{S}) & \text{if } \mathbf{x} \in \mathcal{S} \\ -d(\mathbf{x}, \partial\mathcal{S}) & \text{if } \mathbf{x} \in \mathcal{S}^c \end{cases}$$

- Concretely, f's 0-level set should represent the border of $\mathcal{S}$.
- We can then use Marching Cubes [4] or similar isoextraction algorithms to obtain a mesh

NCH Basis Functions

- Define the NCH basis functions as:

$$f_i^+(\mathbf{x}) = \langle \mathbf{n_i}, \mathbf{x} - \mathbf{p_i} \rangle - \frac{1}{2r_i^+} ||\mathbf{x} - \mathbf{p_i}||_2^2$$

  These are:
  - 0 at the sphere/plane's border
  - Positive inside, achieving the maximum at $\mathbf{c_i}$
  - Negative outside
  - Gradient matches $\mathbf{n_i}$ at $\mathbf{p_i}$
- $f_i^-(\mathbf{x})$ equivalently, with the normal pointing outwards
- We call $f_i(\mathbf{x}) = \frac{1}{2}(f_i^+(\mathbf{x}) - f_i^-(\mathbf{x}))$ the Symmetric NCH SDF

NCH Signed Distance Function (SDF)

- For each variant, its NCH SDF is:

$$f(\mathbf{x}) = \max_{1 \leq i \leq N} f_i(\mathbf{x})$$

- Intersection of complements of half-spaces and balls
- Fits the surface at every point
- Respects the normal vectors
- Isoextraction on its 0-level set outputs the reconstruction.

Naïve Non-Convex Hull

- Finds $r_i = \frac{1}{2\rho_i}$ for each point as explained above
- $\mathcal{O}(N^2)$, where N is the number of points in the cloud
- Extremely slow for large point clouds

Fitting Algorithm

for $i \in 1 \dots N$ **do**
    $\rho_i \leftarrow 0$
    for $j \in 1 \dots N$ **do**
        if $i \neq j$ **then**
            $a \leftarrow \langle \mathbf{n_i}, \mathbf{p_j} - \mathbf{p_i} \rangle$
            $b \leftarrow ||\mathbf{p_j} - \mathbf{p_i}||^2$
            if $a > \rho_i b$ **then**
                $\rho_i \leftarrow \frac{a}{b}$
return $\{\rho_1, \dots, \rho_N\}$

SDF Evaluation

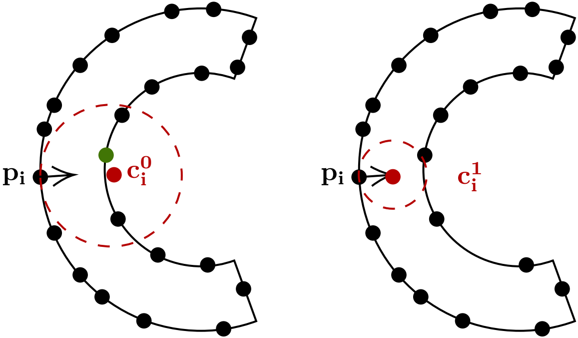
$f_x \leftarrow -\infty$
for $i \in 1 \dots N$ **do**
    $a \leftarrow \langle \mathbf{n_i}, \mathbf{x} - \mathbf{p_i} \rangle - \rho_i ||\mathbf{x} - \mathbf{p_i}||^2$
    if $a > f_x$ **then**
        $f_x \leftarrow a$
return f_x

Shrinking Ball (Ma, Bae, and Choi [5])

- Computes r_i via iterative refinement:
  - Begin with a heuristic estimate for r_i
  - Find the nearest neighbor p_t of c_i with the current estimate
  - Attempt to shrink the ball further by using p_t
  - Repeat until convergence
- Expected worst case $\mathcal{O}(N \log N)$ using a Kd-Tree, as the number of shrinking steps are bounded

for $i \in 1 \ldots N$ **do**
    $r_i \leftarrow \text{InitialRadius}(i)$
    for $t \in 1 \ldots t_{\max}$ **do**
        $j \leftarrow \text{NearestNeighbor}(\mathbf{p_i} + r_i \mathbf{n_i}, \mathbf{p_i})$ {Search excludes $\mathbf{p_i}$}
        $r_i' \leftarrow \text{TangentSphereRadius}(\mathbf{p_i}, \mathbf{p_j})$
        $\text{swap}(r_i, r_i')$
        if $|r_i - r_i'| < \delta$ **then**
            break {Algorithm has converged}
    $\text{AdjustRadiusHeuristic}(i)$
return $\{r_1, \ldots, r_N\}$

Shrinking Ball: Intuition



Shrinking Ball: Issues for NCH Estimation

- Shrinking Ball is meant for MAT approximation, not NCH
- Suffers from false convergence issues
- Needs to use exact Kd-Trees, approximates wreak havoc on estimation

Contribution: Adapting for NCH Approximation

Contribution: Allowing Approximate Kd-Trees

Contribution: Fixing False Convergence

Examples

Performance Comparison

Conclusions

- Addressed issues with the Shrinking Ball (SB) algorithm that prevented it from being used for surface reconstruction
- Improved the complexity of fitting the Non-Convex Hull from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$
- Proposed accuracy improvements to the algorithm, which we call Shrinking Planes (SP)
- Evaluated the changes and found a sizable improvement in the approximation quality of SP with respect to SB.

Future Work

- Isoextraction speed
  - $O(N)$ to evaluate SDF
  - We have some unpublished improvements, but not asymptotic
- Noise robustness
  - Currently extremely sensitive
  - Noise in the normal vectors are a challenge
- Radius initialization functions
  - Published ones are wrong, don't parallelize, or are a non-solution
- GPU implementation
- Convergence properties

Bibliography I

Y. Zhu, F. Sun, Y.-K. Choi, B. Jüttler, and W. Wang, "Computing a compact spline representation of the medial axis transform of a 2d shape", *Graph. Models*, vol. 76, no. 5, pp. 252–262, Sep. 2014, ISSN: 1524-0703. DOI: 10.1016/j.gmod.2014.03.007.

H. Blum, "A Transformation for Extracting New Descriptors of Shape", *Models for the Perception of Speech and Visual Form*, W. Wathen-Dunn, Ed., pp. 362–380, 1967.

G. Taubin, "Non-convex hull surfaces", in *SIGGRAPH Asia 2013 Technical Briefs*, ser. SA '13, Hong Kong, Hong Kong: ACM, 2013, 2:1–2:4, ISBN: 978-1-4503-2629-2. DOI: 10.1145/2542355.2542358.

W. E. Lorensen and H. E. Cline, "Marching cubes: A high resolution 3d surface construction algorithm", *SIGGRAPH Comput. Graph.*, vol. 21, no. 4, pp. 163–169, Aug. 1987, ISSN: 0097-8930. DOI: 10.1145/37402.37422.

J. Ma, S. W. Bae, and S. Choi, "3d medial axis point approximation using nearest neighbors and the normal field", *The Visual Computer*, vol. 28, no. 1, pp. 7–19, Jan. 2012, ISSN: 1432-2315. DOI: 10.1007/s00371-011-0594-7.