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Non-Convex Hull Surfaces

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Typical Surface Reconstruction Pipeline

Oriented Points → Reconstruction Method → Surface Representation

Positions & Normals

Watertight Surface → Polygon Mesh

Interpolating Polygon Meshes [Combinatorial]

- Boissonnat [1984] (Natural Neighbor Interpolation)
- Edelsbrunner [1984] (Alpha Shapes)
- Amenta et al. [1998,2001] (Power Crust)
- Bernardini et al. [1999] (Ball Pivoting)
- Dey [2007] (Book)
- others

Implicit Function [Optimization]

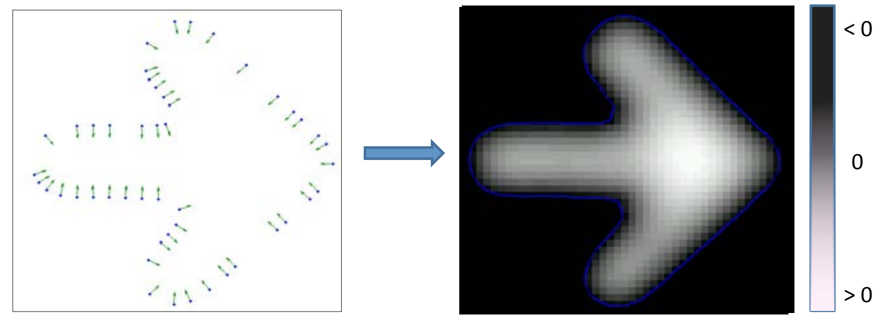
- Hoppe et al. [1992]
- Curless et al [1996]
- Whitaker [1998]
- Carr et al.[2001] (RBFs)
- Davis et al. [2002]
- Ohtake et al. [2004] (MPI)
- Turk et al. [2004]
- Shen et al. [2004]
- Sibley-Taubin [2005]
- Calakli-Taubin [2011] (SSD)

Poisson Surface Reconstruction

Kazhdan et al. [2006]

Manson et al. [2008]

Implicit Surface Reconstruction

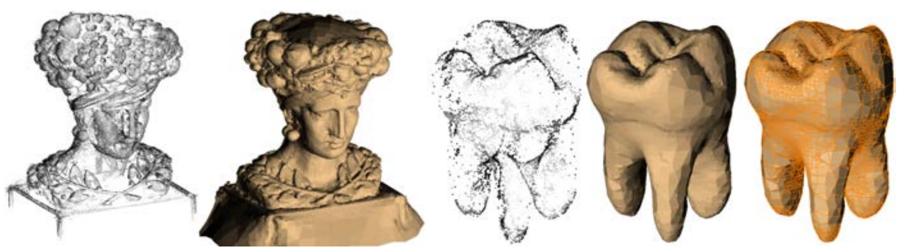


Oriented Points, ***D***
(samples from unknown surface ***S***)

Computed Implicit Surface, ***S'***

Find a scalar valued function **f(p)**, whose zero level set **S'={p:f(p)=0}** is the estimate for true surface ***S***

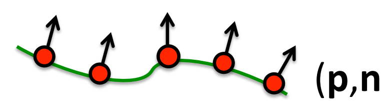
Smooth Signed Distance Surface Reconstruction
[Calakli & Taubin 2011]



$$E(f) = \sum_{i=1}^{N} f(\mathbf{p}_i)^2 + \lambda_1 \sum_{i=1}^{N} \left\| \nabla f(\mathbf{p}_i) - \mathbf{n}_i \right\|^2 + \lambda_2 \int_V \| \mathrm{H}f(\mathbf{x}) \|^2 \, d\mathbf{x}$$

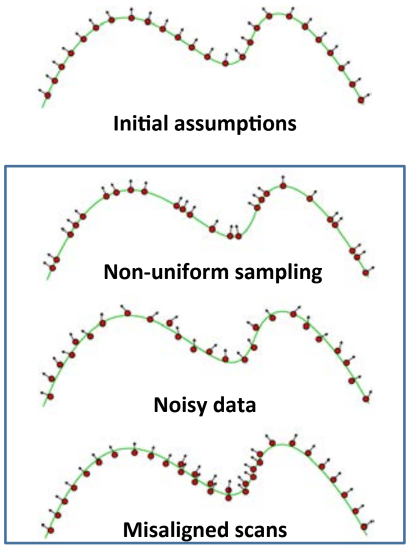
Particularly Good at Extrapolating Missing Data

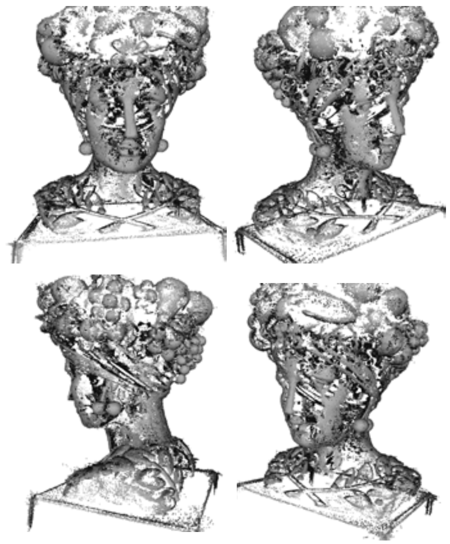
Oriented Points as Surface Samples



- Oriented point **(p,n)**
- **p**: sample of surface location
- **n**: sample of surface normal vector @ **p**
- **Initial Assumptions**
- Surface is bounded, oriented, manifold, and without boundary
- Samples are dense and uniformly distributed
- Normal vectors are unit length and consistently oriented towards the "outside"
- Low Noise

Real Data Sets Present Additional Challenges





Reconstruction with an Approximate Signed Distance

- Input: oriented points contained in a bounding volume V

  $D = \{ (\mathbf{p}_i, \mathbf{n}_i) \; i=1,\ldots,N \}$
- Output: implicit surface

  $S = \{ \mathbf{x} \mid f(\mathbf{x}) = 0 \}$

  with the function defined on V, such that

  $f(\mathbf{p}_i) = 0$ and $\nabla f(\mathbf{p}_i) = \mathbf{n}_i$ for $i=1,\ldots,N$

  $|\nabla f(\mathbf{p}_i)|=1$
- Family of implicit functions parameterized by a finite number of parameters.
- Estimate parameters so that the conditions are satisfied, if not exactly, then in the least-squares sense.
- Partition V into a volumetric mesh M, such as a voxel grid or dual octree.
- Evaluate approximate signed distance on M-vertices, and compute isosurface.

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NCH Surface Reconstruction Algorithm

- Produces interpolatory implicit surface

  $f(\mathbf{p}_i) = 0$ and $\nabla f(\mathbf{p}_i) = \mathbf{n}_i \quad \forall (\mathbf{p}_i,\mathbf{n}_i) \in D$
- Isosurface approximates
- Defined by one parameter per point
- Analytic, direct, non-iterative algorithms to estimate parameters and to evaluate function
- Can be implemented in a few lines of code
- Produces high quality surfaces
- **Simple and Elegant**
- On the negative side: $O(n^2)$ not scalable
- But of practical use

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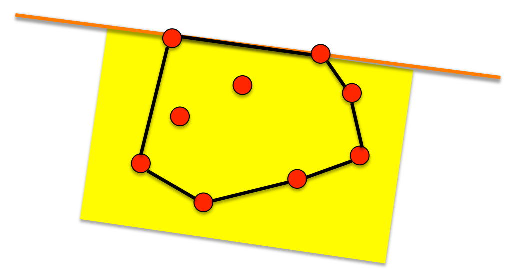
Convex Hull of a Set of Points



- Smallest convex set containing all the points
- Equal to the intersection of all the **Supporting linear half-spaces**

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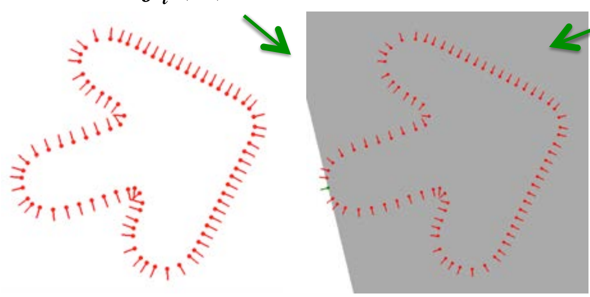
Oriented Convex Hull

One Linear function per point $f_i(x) = n_i^t(x - p_i)$

$f_i(x) > 0$ $f_i(x) < 0$

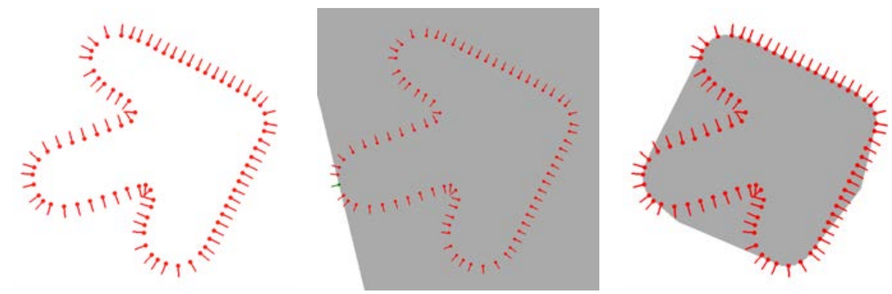


Supporting Linear Half Space if $f_i(p_j) \le 0 \quad \forall j \ne i$

Not every point defines a Supporting Linear Half Space

Oriented Convex Hull Surface

Boundary of the Intersection of all Supporting Linear Half Spaces

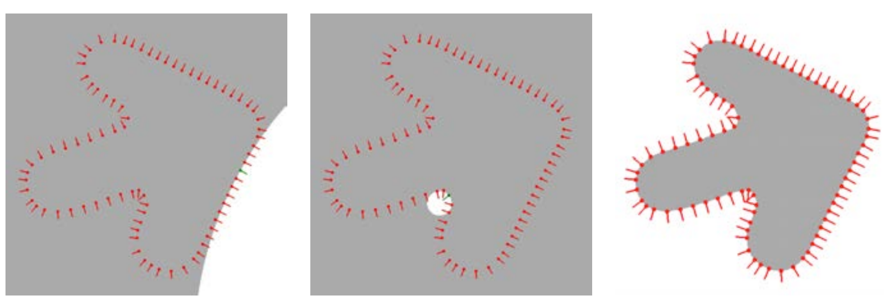


$S = \{x : f(x) = 0\} \qquad f(x) = \max_i f_i(x)$

Non-Convex Hull

Spherical Half Space = Space outside of a Sphere

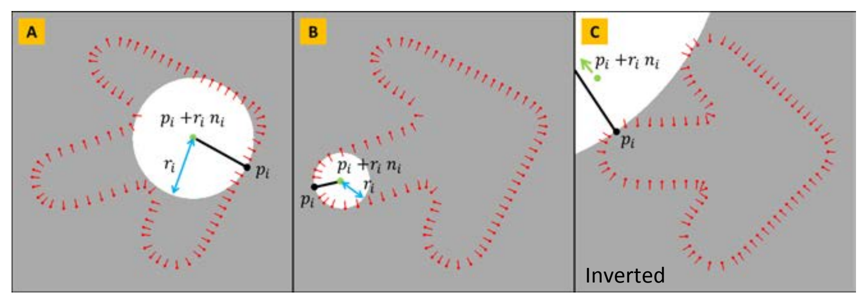
Intersection of all Supporting Spherical Half Spaces



$f(x) = \max_i f_i(x)$

But now every point has a supporting function

Geometry of the Support Functions



$q_i = p_i + r_i n_i$ Max r_i so that $f_i(p_j) \le 0$ for all j

$$f_i(x) = \frac{1}{2r_i}\left(r_i^2 - \|x - q_i\|^2\right) \qquad f_i(x) = n_i^t(x - p_i) - \frac{1}{2r_i}\|x - p_i\|^2$$

Properties of Non-Convex Hull Function

$$f(x) = \max_i f_i(x)$$

$$f_i(x) = n_i^t(x - p_i) - \frac{1}{2r_i}\|x - p_i\|^2$$

Max r_i so that $f_i(p_j) \le 0$ for all j

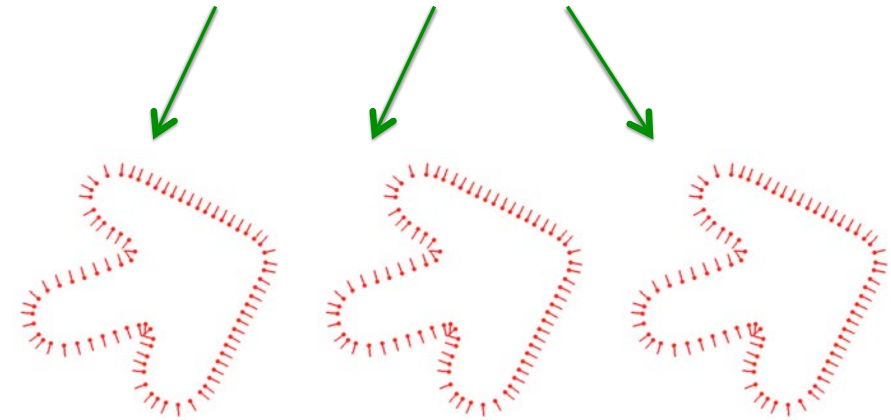
$$f_i(p_i) = 0 \qquad \nabla f_i(p_i) = n_i$$

$$f_j(p_i) \le 0 = f_i(p_i) \Rightarrow f(p_i) = f_i(p_i) = 0$$

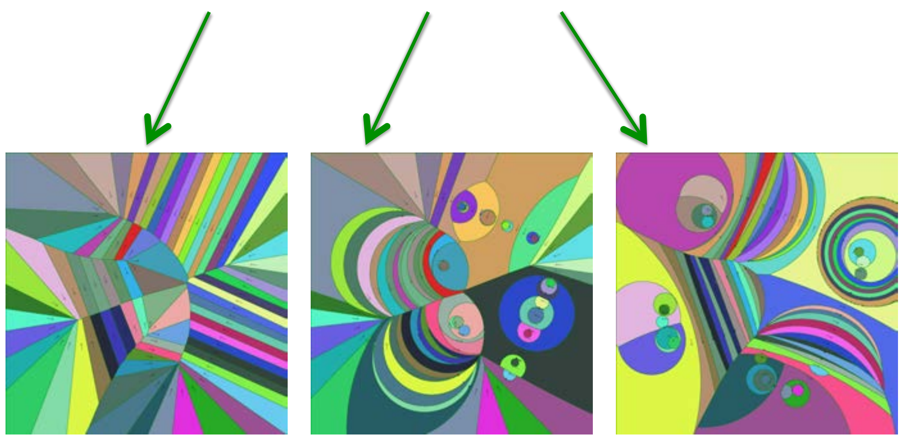
$p_i \in F_i = \{x : f_i(x) > f_j(x) \forall j \ne i\}$ is open

$$\Rightarrow \nabla f(p_i) = \nabla f_i(p_i) = n_i$$

Voronoi and Apollonious Diagrams



Voronoi and Apollonious Diagrams



$$F_i = \{x : f_i(x) > f_j(x) \ \ \forall j \ne i\}$$

NCH Surface Reconstruction

$$f_i(x) = n_i^t(x - p_i) - \rho_i\|x - p_i\|^2 \qquad \rho_i = \frac{1}{2r_i} \qquad 0 \le \rho_i < \infty$$

$$\rho_i = \min\left\{\frac{n_i^t(p_j - p_i)}{\|p_j - p_i\|^2} : j \in J_i\right\} \qquad J_i = \left\{j : n_i^t(p_j - p_i) > 0\right\}$$

But $\rho_i = 0$ if $J_i = \varnothing$

```
procedure estimateNCH () {
    for i = 1 to i = N step 1 do {
        ρ_i = 0
        for j = 1 to j = N step 1 do {
            if j = i continue
            a = n_i^t (p_j − p_i)
            b = ||p_j − p_i||^2
            if ( a − ρ_i b > 0 ) ρ_i = a/b
        }
    }
}
```

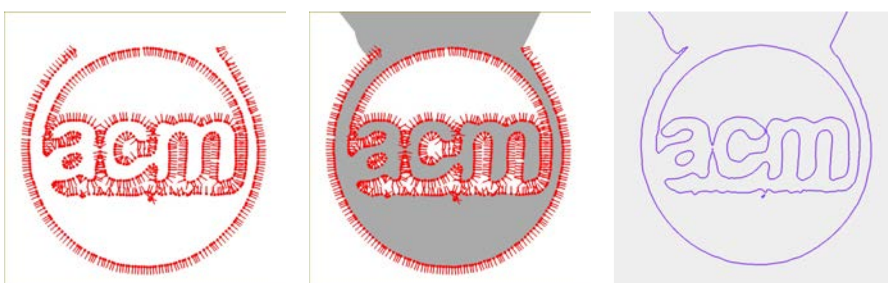
$O(n^2)$ but $O(n)$ with $O(n)$ processors


```
procedure evaluateNCH (x) {
    f_x = −∞
    for i = 1 to i = N step 1 do {
        a = n_i^t (x − p_i)
        b = ||x − p_i||^2
        c = a − ρ_i b
        if ( c > f_x ) f_x = c
    }
    return f_x
}
```

$f_i(x)$

$f(x) = \max_i f_i(x)$

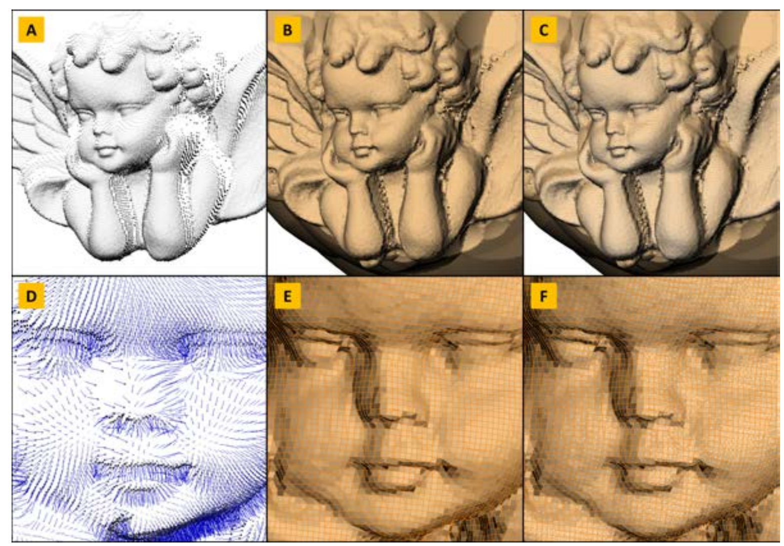
Another 2D Result



$C = \{x : f(x) = 0\}$

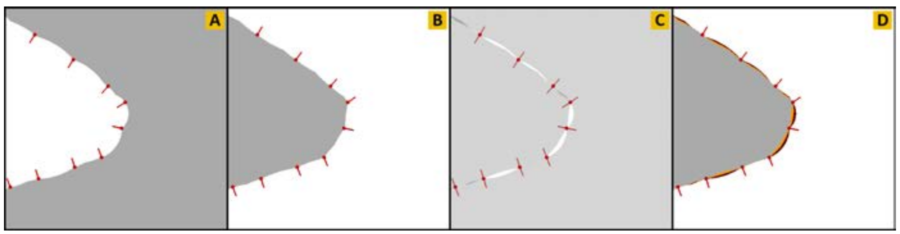
Function evaluated on pixel grid and isocurve computed

A 3D Example [Regular Voxel Grid 500^3]



Symmetric Non-Convex Hull

- If orientation of normal vectors is reversed, a different NCH Function results.
- Compute $f_i^+(x)$ from $\{(p_i, n_i) : i = 1, \ldots, N\}$
- Compute $f_i^-(x)$ from $\{(p_i, -n_i) : i = 1, \ldots, N\}$
- Define $f(x) = \{f_i^+(x) - f_i^-(x)\}/2$



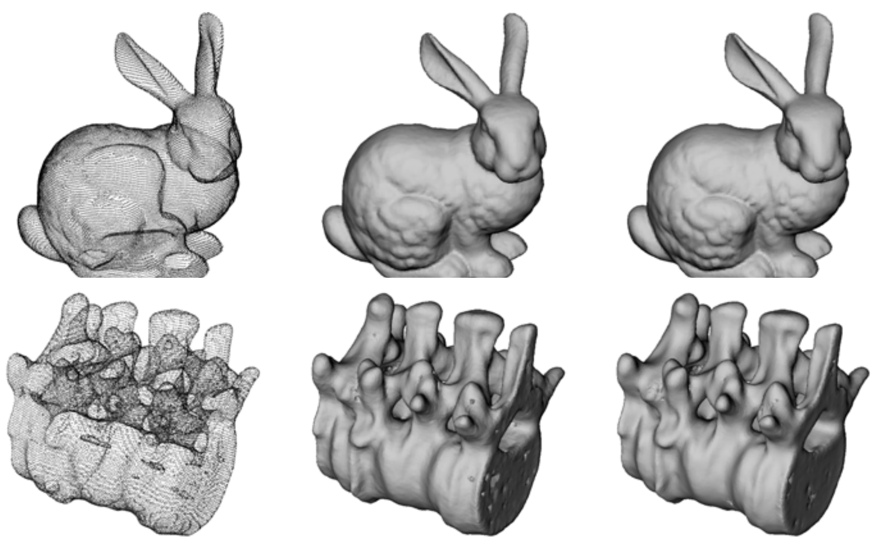
NCH Surface Representation

- Set of oriented points with two additional scalar attributes

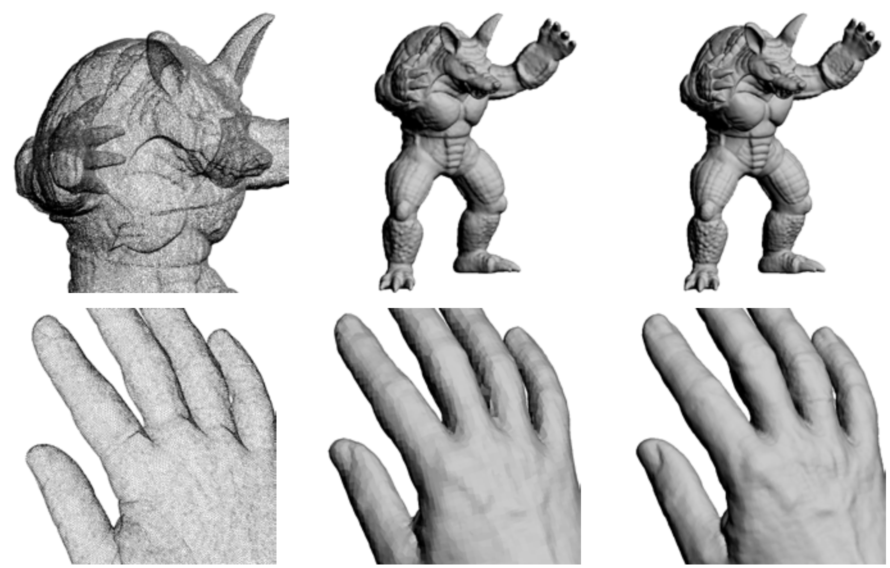
$$\{(p_i, n_i, \rho_i^+, \rho_i^-) : i = 1, \ldots, N\}$$

- Can be saved as a PLY file
- Evaluate on tet-mesh vertices and compute piece-wise-linear isosurface
- Evaluate on dual vertices of octree and run Dual Marching Cubes

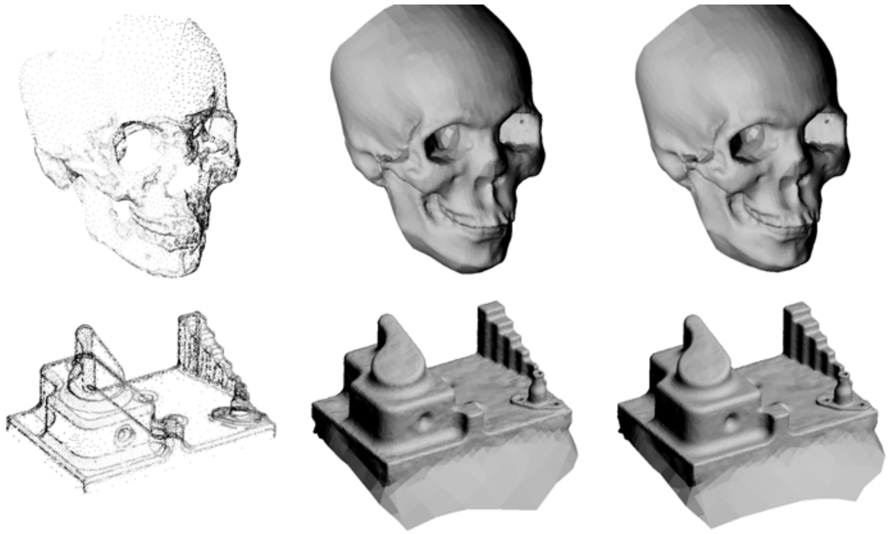
3D Results: evenly sampled low noise



3D Results: evenly sampled low noise



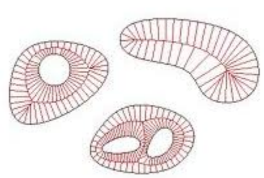
3D Results: unevenly sampled low noise



3D Results: unevenly sample and noise

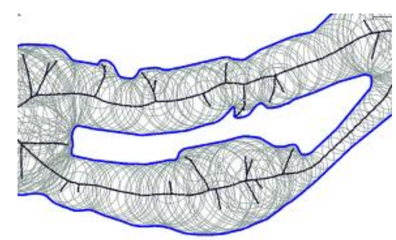
Relation to the Medial Axis Transform

- The finite set of oriented points is replaced by the continuous boundary surface S of a bounded solid object O, which is an open set in 3D
- The surface S is smooth, with a continuous unit length normal field pointing towards the inside of O, and continuous curvatures.



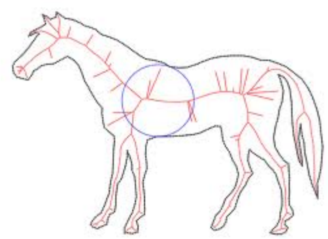
Relation to the Medial Axis Transform

- A medial ball of O is an open ball contained in O which is maximal with respect to inclusion
- The Medial Axis Transform of O is the family MAT(O) of medial balls of O.



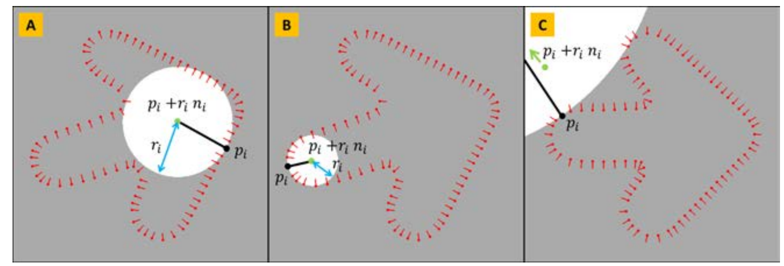
Relation to the Medial Axis Transform

- The Medial Axis of O, denoted MA(O), is the set of centers of medial balls
- Since for each medial axis point there is a unique medial ball, the MAT(O) can also be represented as a set of center-radius pairs (q,r)



Relation to the Medial Axis Transform

- Our construction is an alternative representation of MAT(O) as a list of center-vector-radius tuples (p,n,r), where each medial ball is specified by one of its boundary points, the unit length vector that points to the center of the ball, and the radius.



Questions?

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