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EDUARDO DUBUC
GABRIEL TAUBIN

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ANALYTIC RINGS

by Eduardo DUBUC and Gabriel TAUBIN

INTRODUCTION.

The C-algebras that are considered in the practice of analytic geometry (meaning the theory of several complex variables) all carry a richer structure. We develop here the concept of *analytic ring*, which takes into account explicitly this extra structure. On the other hand, analytic rings will be essential for the construction of models of Synthetic Differential Geometry well adapted to the study of complex manifolds and analytic varieties. Thus, we study here a class of C-algebras with the richer structure given by operations associated not only to polynomials but also to all holomorphic functions. If a holomorphic function is defined only in an open set $U \subset C^n$, the corresponding operation will be a partial operation.

An *analytic ring* in a category $\mathcal{E}$ is defined as a functor $\mathcal{C} \to \mathcal{E}$, defined over the category $\mathcal{C}$ of open subsets of C^n, $n \in N$, and holomorphic functions. This functor is required to preserve all *transversal pullbacks* that may exist in $\mathcal{C}$.

Examples of analytic rings are the following: rings of holomorphic functions on an open set of C^n, or more generally, on any complex manifold. Local rings of germs of holomorphic functions, as well as analytic algebras in the sense of Malgrange [7]. In particular, all complex Weil algebras. Rings of sections of any analytic space. The sheaf of continuous complex valued functions on any topological space X is an analytic ring in the topos of sheaves over X. As well as the structure sheaf of any analytic space. Also, the inclusion from $\mathcal{C}$ into the category of analytic spaces is an analytic ring in this category.

This article is divided in three parts. In the first we give the definition of analytic rings and some of their basic properties, and we treat the local analytic rings. In the second we consider the analytic spaces.

Finally, in the third, we construct the generic analytic ring in a category with finite limits. That is, the (algebraic) theory of analytic rings. We include a Section 0 with some background material. Also, we include a summary preceding the text.

SUMMARY.

DEFINITION 1.1. An *analytic ring* in a category $\mathcal{E}$ is a functor $F: \mathcal{C} \to \mathcal{E}$ defined on the category $\mathcal{C}$ of open subsets of complex euclidean spaces and holomorphic functions. This functor is required to preserve the terminal object and all transversal pullbacks. Morphisms of analytic rings are natural transformations.

It follows that the underlying functor $F \mapsto F(\mathrm{C})$ is faithful and that $F(\mathrm{C})$ is a C-algebra. By abuse of notation we write F for $F(\mathrm{C})$; then, for each open $U \subset \mathrm{C}^n$, there is a well distinguished subobject of n-tuples $F(U) \subset F^n$ where the partial «U-ary» operations corresponding to holomorphic functions $U \to \mathrm{C}$ are defined.

EXAMPLES. The following is a list of C-algebras which are (the underlying C-algebras of) analytic rings. First, in the category $\mathcal{E}ns$ of sets:

(1) $\mathcal{O}_n(U)$ = holomorphic functions $U \to \mathrm{C}$, for any open $U \subset \mathrm{C}^n$.

(2) $\mathcal{O}_{n,p}$ = germs of holomorphic functions in n variables at $p \in \mathrm{C}^n$.

(3) Analytic algebras in the sense of Malgrange, i. e. quotients $\mathcal{O}_{n,p}/I$, where I is any ideal.

In the category $Sh(X)$ of sheaves on a topological space X:

(4) C_X = sheaf of germs of continuous complex-valued functions on X.

(5) $\mathcal{O}_U$ = sheaf of germs of holomorphic functions on an open set $X = U \subset \mathrm{C}^n$.

(6) $\mathcal{O}_X$ = structure sheaf of any analytic space X.

In the category of analytic spaces, or the more general A-ringed spaces defined below:

(7) $(\mathrm{C}, \mathcal{O}_\mathrm{C})$ = the complex numbers with the sheaf of germs of holomorphic functions.

Let $\mathcal{E}$ be a category with finite limits and commuting filtered colimits, and let $\mathcal{A}n(\mathcal{E})$ be the category of analytic rings in $\mathcal{E}$.

PROPOSITION 1.3. The category $\mathcal{A}n(\mathcal{E})$ has finite limits and commuting filtered colimits, and they are computed pointwise.

Note that Example (1) is the representable functor $\mathcal{C}(U, -)$, and Example (2) follows from it as a filtered colimit in $\mathcal{A}n(\mathcal{E}_{n\Delta})$.

DEFINITION 1.8. Given any two functors $F, G: \mathcal{C} \to \mathcal{E}$, a natural transformation $G \to F$ *is local* if for each open inclusion $U \subset V$ in $\mathcal{C}$,

$$\begin{array}{ccc} G(U) & \longrightarrow & G(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(V) \end{array}$$

is a pullback in $\mathcal{E}$.

EXAMPLE 1.12. If $\phi: F \to \mathbf{C}$ is local in $\mathcal{A}n(\mathcal{E}_{n\Delta})$, then the $(\mathbf{C}-\{0\})$-ary operation $1/z$ is defined at $f \in F$ iff $\phi(f) \neq 0$ in $\mathbf{C}$. Thus F is in particular a local $\mathbf{C}$-algebra.

MAIN THEOREM 1.10. With the notations of 1.8, if G is product preserving and F is an analytic ring which preserves open covers, then G is an analytic ring (and preserves open covers).

The basic tool to prove this theorem is the *inverse function Theorem.*

THEOREM 1.18. If $I \subset \mathcal{O}_{n,p}$ is any ideal, then the quotient $\mathbf{C}$-algebra $\mathcal{O}_{n,p}/I$ has a unique structure of analytic ring such that $\pi: \mathcal{O}_{n,p}/I \to \mathbf{C}$ becomes a local morphism.

To prove this theorem, we use the Main Theorem and the *Fermat Property*: if h is a holomorphic function of n variables, then

$$h(z) - h(x) = \Sigma_i (z_i - x_i) h_i(x, z).$$

Example (3) follows from this theorem.

PROPOSITION 2.1. The functor

$$Open(X)^{op} \times \mathcal{C} \to \mathcal{E}_{n\Delta} \text{ defined by } (H, U) \mapsto Continuous(H, U)$$

defines an analytic ring $\mathbf{C}_X$ in $Sh(X)$ (Example (4)), which preserves open covers.

DEFINITION 2.3. An *A-ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is an analytic ring in $Sh(X)$ furnished with a local morphism $\rho_X : \mathcal{O}_X \to C_X$ (it follows that ρ_X is unique). Morphisms of A-ringed spaces are defined as usual.

PROPOSITION 2.7. If $V \in \mathcal{C}$, the functor

$$Open(V)^{op} \times \mathcal{C} \to \mathcal{E}ns \quad \text{defined by} \quad (H, U) \mapsto Holomorphic(H, U)$$

defines an analytic ring $\mathcal{O}_V$ in $Sh(V)$ (Example (5)). The inclusion $\mathcal{O}_V \subset C_V$ is local, and the pair $(V, \mathcal{O}_V)$ is an A-ringed space.

Let $\mathcal{A}$ be the category of A-ringed spaces and $i : \mathcal{C} \to \mathcal{A}$ the functor defined by $U \mapsto (U, \mathcal{O}_U)$.

THEOREM 2.8. For any A-ringed space $(X, \mathcal{O}_X)$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad i \quad} & \mathcal{A} \\ {\scriptstyle \mathcal{O}_X}\downarrow & & \downarrow{\scriptstyle \mathcal{A}((X, \mathcal{O}_X), -)} \\ Sh(X) & \xrightarrow{\quad \Gamma \quad} & \mathcal{E}ns \end{array}$$

(Γ = global sections).

COROLLARY 2.9. The functor $i : \mathcal{C} \to \mathcal{A}$ is an analytic ring in $\mathcal{A}$ (Example (7)).

THEOREM 2.10. If $\mathcal{J} \subset \mathcal{O}_V$ is any sheaf of ideals, then the pair $(E, \mathcal{O}_E)$ is an A-ringed space, where $E = Zeros(\mathcal{J}) \subset V$ and $\mathcal{O}_E$ is the quotient sheaf $\mathcal{O}_V / \mathcal{J}$ restricted to E. It follows that all analytic spaces are A-ringed spaces (Example (6)).

REMARK 3.7. Let $\mathcal{C} \to \mathcal{A}n$ be the *generic analytic ring.* It follows from Corollary 2.9 that there is a finite limit preserving functor $Spec : \mathcal{A}n \to \mathcal{A}$. We do not know if this functor is full and faithful, as it is the case with the corresponding functor in the algebraic and C^∞ situations.

0. GENERAL RESULTS ON THEORY OF CATEGORIES AND HOLOMORPHIC FUNCTIONS.

About limits. We will call *limit* a projective limit, and *colimit* the dual concept.

0.1. PROPOSITION. *Let $\mathfrak{A}$ be an arbitrary category, then the following statements are equivalent:*

i) $\mathfrak{A}$ has finite limits.

ii) $\mathfrak{A}$ has terminal object ($= 1$), finite products and equalizers.

iii) $\mathfrak{A}$ has pullbacks and terminal object.

0.2. COROLLARY. Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories with finite limits and $F : \mathfrak{A} \to \mathfrak{B}$ a functor; then the following statements are all equivalent:

i) F preserves finite limits.

ii) F preserves terminal object, finite products and equalizers.

iii) F preserves pullbacks and terminal object.

0.3. PROPOSITION. *In a category $\mathfrak{A}$, the following statements are equivalent:*

i)

$$\begin{array}{ccc} L & \xrightarrow{p_1} & W \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

is a pullback.

ii)

$$\begin{array}{ccc} L & \xrightarrow{\phi} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times W & \xrightarrow{f \times g} & U \times V \end{array}$$

is a pullback, where $\phi : L \to V$ is the composite $gp_1 = fp_2$ and $p = (p_1, p_2)$.

0.4. COROLLARY. In any category $\mathfrak{A}$ the following are equivalent:

i) $\phi : A \to B$ is a monomorphism.

ii)

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle id_A}\downarrow & & \downarrow{\scriptstyle \phi} \\ A & \xrightarrow{\phi} & A \end{array}$$

is a pullback.

iii)

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ A \times A & \xrightarrow{\phi \times \phi} & B \times B \end{array}$$

is a pullback.

0.5. COROLLARY. If a functor $F: \mathfrak{A} \to \mathfrak{B}$ preserves pullbacks, then it preserves monomorphisms.

About ring objects.

0.6. PROPOSITION *(Unicity of the inverse). Let $\mathcal{E}$ be a category and A a ring object in $\mathcal{E}$, then*

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle (x,\,-x)}\downarrow & & \downarrow{\scriptstyle 0} \\ A\times A & \xrightarrow{x_1+x_2} & A \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle (x,\,x)}\downarrow & & \downarrow{\scriptstyle 0} \\ A\times A & \xrightarrow{x_1-x_2} & A \end{array}$$

$$\begin{array}{ccc} Units(A) & \xrightarrow{(x,\,x^{-1})} & A\times A \\ \downarrow & & \downarrow{\scriptstyle x_1x_2} \\ 1 & \xrightarrow{1} & A \end{array}$$

are pullback diagrams.

0.7. PROPOSITION. *Let $\mathcal{E}$ be a category with terminal object and finite products, A a ring object in $\mathcal{E}$ and V any subobject of A. Let $i: V \rightarrowtail A$ be the inclusion monomorphism. Then the following are equivalent:*

i)

$$\begin{array}{ccc} E & \xrightarrow{p_2} & W \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

is a pullback.

ii)

$$E \xrightarrow{p} U\times W \overset{h}{\underset{0}{\rightrightarrows}} A$$

is an equalizer, where $p = (p_1, p_2)$ and $h = if - ig$.

About categories of fractions.

0.8. PROPOSITION. *Let $\mathcal{C}$ be a category with finite limits, $G_a: \mathcal{C} \to \mathfrak{D}_a$ $(a \in \Lambda)$ a family of limit preserving functors and*

$$\Sigma = \{ \sigma \in arrow(\mathcal{C}) \mid G_a(\sigma) \text{ is invertible in } \mathfrak{D}_a, \forall a \in \Lambda \}$$

Then Σ admits a calculus of right fractions and the following holds:

i) The category of fractions $\mathcal{C}[\Sigma^{-1}]$ has finite limits and the canonical functor $p_\Sigma: \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves those limits.

ii) The family of functors $H_\alpha : \mathcal{C}[\Sigma^{-1}] \to \mathfrak{D}_\alpha$ *defined by the equations:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;p_\Sigma\;} & \mathcal{C}[\Sigma^{-1}] \\ & {\scriptstyle G_\alpha}\searrow \quad \swarrow{\scriptstyle H_\alpha} & \\ & \mathfrak{D}_\alpha & \end{array}$$

$G_\alpha = H_\alpha \circ p_\Sigma$ *is conservative (i. e.,* $H_\alpha(\tau)$ *isomorphism for each* $\alpha \in \Lambda$ *implies* τ *isomorphism), and* H_α *preserves finite limits for each* $\alpha \in \Lambda$.

PROOF. Cf. Gabriel & Zisman [4].

About holomorphic functions.

0.9. PROPOSITION. *Let* $V = V_1 \times \ldots \times V_n$, $V_i \subset \mathbb{C}$ *be open subsets of* $\mathbb{C}$, *and* $g : V \to \mathbb{C}$ *a holomorphic function, then there exists one and only one holomorphic function* $h : V \times V_1 \to \mathbb{C}$ *such that the following equation holds for all*

$$(x_1, \ldots, x_n) \in V \quad \text{and} \quad z_1 \in V_1 :$$

$$g(x_1, \ldots, x_n) - g(z_1, x_2, \ldots, x_n) = (x_1 - z_1)\, h(x_1, \ldots, x_n, z_1).$$

PROOF. Let $\phi : V \times V_1 \to \mathbb{C}$ be defined by

$$\phi(x_1, \ldots, x_n, z_1) = g(x_1, \ldots, x_n) - g(z_1, x_2, \ldots, x_n).$$

ϕ is holomorphic because g is holomorphic. We define $h : V \times V_1 \to \mathbb{C}$ by

$$h(x_1, \ldots, x_n, z_1) = \begin{cases} \dfrac{\phi(x_1, \ldots, x_n, z_1)}{x_1 - z_1} & \text{if } x_1 \neq z_1 \\[2ex] \dfrac{\partial g}{\partial x_1}(x_1, \ldots, x_n) & \text{if } x_1 = z_1 . \end{cases}$$

It is easy to see that h is continuous in $V \times V_1$ and holomorphic in

$$V \times V_1 - \{ (x_1, \ldots, x_n, z_1) \mid x_1 = z_1 \}.$$

It suffices to see that h is holomorphic in each variable (Gunning & Rossi [6], Theorem 2, page 2). It is easy to see that h is holomorphic in the variables $x_2, \ldots, x_n$. The proof that h is holomorphic in the variable x_1 is equivalent to the proof that h is holomorphic in the variable z_1: because the definition of h is symmetric in x_1 and z_1. We fix $x_1, \ldots, x_n$; $h(x_1, \ldots, -) : V_1 \to \mathbb{C}$ is continuous and holomorphic in $V_1 - \{x_1\}$, then it is holomorphic in V_1 (Ahlfors [1], Theorem 7, page 124). The unicity is an evident fact, because any two solutions coincide for $x_1 \neq z_1$, and therefore by continuity they also coincide for $x_1 = z_1$.

0.10. COROLLARY. With the same hypothesis as in the Proposition 0.9 there exist holomorphic functions $h_1, \dots, h_n : V \times V \to \mathbb{C}$ such that the following equality is true in $V \times V$:

$$g(x_1, \dots, x_n) - g(z_1, \dots, z_n) = \sum_{1 \leqslant i \leqslant n} (x_i - z_i) h_i(x_1, \dots, x_n, z_1, \dots, z_n).$$

0.11. REMARK. The fact that Corollary 0.10 does not hold for open sets in general has as consequence that quotients of analytic rings *can not* be computed as the quotient of the underlying C-algebra. However, since the corollary holds for a base of open sets, it will follow that quotients of *local* analytic rings are computed as the quotient of their underlying C-algebra (cf. Theorem 1.18).

0.12. PROPOSITION *(Implicit function Theorem). Let $k \leq n$ be natural numbers, U an open subset of $\mathbb{C}^n$ and $h : U \to \mathbb{C}^k$ a holomorphic function. Let $p \in U$ be such that $h(p) = 0$ and such that the rank of $D(h)(p)$ is k, where*

$$D(h)(p) = \left(\frac{\partial h_j}{\partial x_i}\right)(p)$$

is the Jacobian matrix of h in p. Then there exist open sets $W \subset U$, $p \in W$, $V \subset \mathbb{C}^n$, and a bi-holomorphic bijection $\Psi : V \to W$ such that

$$\Psi(0) = p \quad \textit{and} \quad h \circ \Psi(x) = \pi(x) \textit{ for all } x \in V.$$

Here we have indicated with π the projection into the last k coordinates, $\pi : \mathbb{C}^n \to \mathbb{C}^k$.

About transversal pullbacks.

0.13. DEFINITION. Let M, N and X be complex manifolds, $f : M \to X$ and $g : N \to X$ analytic functions. We say that *f and g are transversal* if, for each $m \in M$ and each $n \in N$ such that $f(m) = g(n) = x \in X$, the images by $f_* : TM \to TX$ and $g_* : TN \to TX$ of the tangent spaces of M at m and of N at n, respectively, generate the tangent space to X in x.

0.14. DEFINITION. Let M be a complex manifold and $h : M \to C^k$ an analytic function; we say that h *is independent*, or that *the components of* h, $h_1, \dots, h_k : M \to C$ *are independent* if for each $p \in M$ such that $h(p) = 0$, the rank of h in p equals k.

Note that $h_1, \dots, h_k$ are independent iff h and the constant function 0 are transversal.

0.15. DEFINITION. We will denote with $\mathcal{O}$ the category of complex manifolds and analytic functions between them. A diagram

$$\begin{array}{ccc} Y & \longrightarrow & N \\ \downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\ f\ } & X \end{array}$$

in $\mathcal{O}$ is a *transversal pullback* if it is a pullback in $\mathcal{O}$ and f and g are transversal.

0.16. DEFINITION. A diagram

$$E \longrightarrow V \underset{0}{\overset{h}{\rightrightarrows}} C^k$$

in $\mathcal{O}$ is an *independent equalizer* if it is an equalizer in $\mathcal{O}$ and the components of h are independent.

0.17. DEFINITION. We denote with $\mathcal{C}$ the category of open subsets of C^n (all n) and holomorphic functions. $\mathcal{C}$ is a full subcategory of $\mathcal{O}$, where all the precedent definitions make sense.

0.18. PROPOSITION. The following statements are equivalent:

i) $\begin{array}{ccc} U & \xrightarrow{\ f\ } & V \\ & & \uparrow{\scriptstyle g} \\ & & W \end{array}$ are transversal, $V \subset C^m$.

ii) $h_1, \dots, h_m$ are independent, where $h : U \times W \to C^m$ is $if - ig$, and where by i we indicate the inclusion of V in C^m.

iii) $\phi_1, \dots, \phi_{2m}$ are independent, where $\phi : U \times W \times V \to C^{2m}$ is defined by

$$\phi(x, y, z) = (f(x) - z,\ g(y) - z).$$

iv) $$\begin{array}{ccc} U\times W & \xrightarrow{f\times g} & V\times V \\ & & \uparrow{\scriptstyle \Delta} \\ & & V \end{array}$$ are transversal.

The proof is straightforward.

0.19. COROLLARY. The following are equivalent:

i)
$$\begin{array}{ccc} E & \xrightarrow{p_2} & W \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

is a transversal pullback.

ii)
$$E \xrightarrow{p} U\times W \overset{h}{\underset{0}{\rightrightarrows}} C$$

is an independent equalizer,

where $p = (p_1, p_2)$ and h is the composition as in the Proposition 0.18.

PROOF. Corollary 0.7 and Proposition 0.18.

0.20. PROPOSITION. *The following are equivalent:*

i) $f: V \to W$ is an independent monomorphism.

ii)
$$\begin{array}{ccc} V & \xrightarrow{id_V} & V \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{f} & W \end{array}$$

is a transversal pullback.

iii)
$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ V\times V & \xrightarrow{f\times f} & W\times W \end{array}$$

is a transversal pullback.

PROOF. Corollary 0.4 and Proposition 0.18.

0.21. PROPOSITION. *Let $F: \mathcal{C} \to \mathcal{E}$ be any functor. Then the following two statements are equivalent:*

i) F preserves transversal pullbacks and terminal object.

ii) F preserves independent equalizers, finite products, terminal object and open inclusions.

PROOF. i) $\Rightarrow$ ii): Any pullback over the terminal object is transversal, so F preserves finite products. An independent equalizer is a transversal pullback to the zero map, so it is preserved by F. Finally, it follows from Proposition 0.20 that F preserves open inclusions.

ii) $\Rightarrow$ i) Let

$$\begin{array}{ccc} E & \xrightarrow{p_2} & W \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

be a transversal pullback; then, by Corollary 0,19,

$$E \xrightarrow{p} U \times W \overset{h}{\underset{0}{\rightrightarrows}} \mathbb{C}^m$$

is an independent equalizer. Thus

$$F(E) \xrightarrow{F(p)} F(U) \times F(W) \overset{0}{\underset{F(h)}{\rightrightarrows}} F(\mathbb{C})$$

is an equalizer; but the inclusion $i: V \to \mathbb{C}^m$ is open, then $F(i)$: $F(V) \to F(\mathbb{C})^m$ is a monomorphism and $F(h): F(U) \times F(W) \to F(\mathbb{C})^m$ is

$$F(h) = F(i)F(f) \cdot F(i)F(g).$$

Then, we deduce from 0.7 that

$$\begin{array}{ccc} F(E) & \xrightarrow{F(p_2)} & F(V) \\ {\scriptstyle F(p_1)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ F(U) & \xrightarrow{F(f)} & F(W) \end{array}$$

is a pullback.

1. THE ANALYTIC RINGS.

1.1. DEFINITION. Let $\mathcal{E}$ be a category with finite limits; a functor F: $\mathcal{C} \to \mathcal{E}$ is an *analytic ring in* $\mathcal{E}$ if any of the two equivalent conditions in Proposition 0.21 holds. A *morphism between analytic rings* is a natural transformation, as functors. We will denote $\mathcal{A}n(\mathcal{E})$ the category just defined. By an abuse of notation, where there will be no danger of confusion, we will write F for $F(\mathbb{C})$ and if $\xi: F \to G$ is a morphism we will write ξ for $\xi_{\mathbb{C}}$.

1.2. OBSERVATION. If $F \in \mathcal{A}n(\mathcal{E})$ and V is an open subset of C^n, then $F(V)$ is a subobject of $F^n = F(C)^n$. This is so because F preserves open inclusions and finite products. Thus an analytic ring may be thought as a C-algebra F, with the additional structure given by partial n-ary operations with domain of definition $F(V) \subset F(C)^n$, one for each holomorphic function $V \to C$, for all V open in some C^n. We will refer to these partial operations as «V-ary operations». We must also note that the forgetful functor $\mathcal{A}n(\mathcal{E}n\Delta) \to \mathcal{E}n\Delta$, which maps each analytic ring F onto its value in C is faithful. Thus each F has an «underline set». We also have a forgetful functor from $\mathcal{A}n(\mathcal{E}n\Delta)$ into the category of C-algebras in $\mathcal{E}n\Delta$, which is faithful but it is not full (as we will see in the Observation 1.7). However in the case of local analytic rings, it is full and faithful (as we will see in Proposition 1.23).

1.3. PROPOSITION. *Let $\mathcal{E}$ be a category with finite limits, then the following statements hold:*

i) $\mathcal{A}n(\mathcal{E})$ has finite limits and they are computed pointwise.

ii) If $\mathcal{E}$ has filtered colimits that commute with finite limits, then $\mathcal{A}n(\mathcal{E})$ has filtered colimits, they are computed pointwise and commute with finite limits.

PROOF. Straightforward.

1.4. PROPOSITION. *For each object U of $\mathcal{C}$, the representable functor $\mathcal{C}(U,-): \mathcal{C} \to \mathcal{E}n\Delta$ preserves all the limits, thus in particular it is an analytic ring in $\mathcal{E}n\Delta$. We denote $\mathcal{O}_n(U)$ the analytic ring $\mathcal{C}(U,-)$, for U an open subset of C^n. We have a functor*

$$\mathcal{C} \to \mathcal{A}n(\mathcal{E}n\Delta)^{op}: U \mapsto \mathcal{O}_n(U),$$

which is full and faithful and it is an analytic ring in $\mathcal{A}n(\mathcal{E}n\Delta)^{op}$. Remark that, in particular, it preserves products; thus

$$\mathcal{O}_{n+m}(U \times V) = \mathcal{O}_n(U) \otimes \mathcal{O}_m(V),$$

where $\otimes$ indicates the coproduct in $\mathcal{A}n(\mathcal{E}n\Delta)$.

PROOF. $\mathcal{A}n(\mathcal{E}n\Delta)^{op}$ is a full subcategory of $(\mathcal{E}n\Delta^{\mathcal{C}})^{op}$, and $U \mapsto \mathcal{O}_n(U)$

is just Yoneda's functor. That it is an analytic ring, when considered with codomain $\mathcal{A}n(\mathcal{E}n_\Delta)^{op}$, follows by a standard argument.

1.5. PROPOSITION. *The forgetful functor* $\mathcal{C} \to \mathcal{E}n_\Delta$ *is an analytic ring in* $\mathcal{E}n_\Delta$*, which we will denote* C by abuse of notation.

1.6. PROPOSITION. *(*$\mathcal{O}_n(U)$ *is the free analytic ring on* $n \mid U$*-generators.) Let* F *be an object of* $\mathcal{A}n(\mathcal{E}n_\Delta)$ *and* $s_1, \dots, s_n$ *elements of* $F = F(\mathrm{C})$ *such that* $(s_1, \dots, s_n) \in F(U)$*; then there exists one and only one morphism of analytic rings*

$$\xi : \mathcal{O}_n(U) \to F \quad \text{such that} \quad \xi(z_i) = s_i \quad (1 \leq i \leq n),$$

where $z_i : U \to \mathrm{C}$ *is the i-th projection* $\mathrm{C}^n \to \mathrm{C}$ *restricted to* U *and* U *is an open subset of* C .

PROOF. It is Yoneda's Lemma, because

$$\mathcal{A}n(\mathcal{E}n_\Delta)[\mathcal{O}_n(U), F] = Nat[\mathcal{C}(U, -), F] \cong F(U).$$

1.7. OBSERVATION. Let $U = \mathrm{C} \times \mathrm{C} - \{0\}$; we know that a function holomorphic in U has a unique holomorphic extension to C^2 (Gunning & Rossi [6], Corollary 6, page 21), and so we have a morphism of C-algebras ev_0: $\mathcal{O}_n(U) \to \mathrm{C}$ («evaluation» in 0). This morphism cannot be a morphism of analytic rings because by Proposition 1.6, any morphism of analytic rings is given by evaluation at a point p of U, and since $0 \notin U$, $p \neq 0$. Clearly $ev_p \neq ev_0$. More generally, let $G \subset \mathrm{C}^n$ be a proper Reinhardt domain (cf. Grauert & Frische [5], Definitions 1.7 and 1.8, pages 5 and 6) and H the complete hull of G (loc. cit., Definition 5.1, page 20). Then we have that any function f holomorphic in G has a unique extension $\hat{f}$ holomorphic in H (loc. cit. Theorem 5.5, page 20). Thus any point of H not in G defines a morphism $\mathcal{O}_n(G) \to \mathrm{C}$ of C-algebras which is not a morphism of analytic rings. Remark that Proposition 1.6 clearly implies Milnor's exercise ([3], Proposition 0.7) with respect to morphisms of analytic rings for any open set G of C^n.

1.8. DEFINITION. Let $\mathcal{E}$ be a category with finite limits, $F, G : \mathcal{C} \to \mathcal{E}$ functors and $\pi : F \to G$ a natural transformation. We say that π is *local* if for all open inclusions $U \subset V$ in $\mathcal{C}$ (including $U = \emptyset$), the square

$$\begin{array}{ccc} F(U) & \longrightarrow & F(V) \\ {\scriptstyle \pi U}\downarrow & & \downarrow{\scriptstyle \pi V} \\ G(U) & \longrightarrow & G(V) \end{array}$$

is a pullback in $\mathcal{E}$. It is equivalent to ask this condition only for $V = \mathbb{C}^n$, all n. This follows from a basic fundamental property of pullback squares that says that given any two composable squares S, T with S a pullback, then the composite square ST is a pullback iff the square T is a pull-

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & T & \downarrow & S & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

back. We have: if $\pi : F \to G$ is local and $l : H \to F$ is any other natural transformation, the composite πl is local iff l is.

We say that a functor $F : \mathcal{C} \to \mathcal{E}$ *preserves a* given covering $U_\alpha \subset U$ in $\mathcal{C}$ if the family $F(U_\alpha) \to F(U)$ is an *universal effective epimorphism* family in $\mathcal{E}$.

1.9. PROPOSITION. *Let $\mathcal{E}$ be a category with finite limits, $F, G : \mathcal{C} \to \mathcal{E}$ functors and $\pi : F \to G$ a local natural transformation. Let:*

i) $\emptyset \longrightarrow U \rightrightarrows V$ *an empty equalizer,*

ii)

$$\begin{array}{ccc} H & \hookrightarrow & W \\ \downarrow & & \downarrow{\scriptstyle h} \\ U & \hookrightarrow & V \end{array}$$

$H = h^{-1}(U)$ a pullback square, with $U \subset V$ an open inclusion,

iii) $U \subset V$ an open inclusion,

iv) $U_\alpha \subset V$ an open covering.

Then, F preserves the equalizer in i), the pullback in ii), the monomorphism in iii) and the covering in iv), provided that G does.

PROOF. iv) and iii) are clear, ii) follows from the composite property of pullback squares mentioned above. Finally, i) is easy.

1.10. THEOREM. *Let $\mathcal{E}$ be a category with finite limits and $F : \mathcal{C} \to \mathcal{E}$ be* *a finite products (and terminal object) preserving functor. Suppose there is a local natural transformation* $\pi : F \to G$, *with* G *an analytic ring in* $\mathcal{E}$ *which preserves open coverings. Then* F *is an analytic ring in* $\mathcal{E}$ *(and it preserves open coverings).*

PROOF. According to Definition 1.1, it only remains to see that F preserves open inclusions and independent equalizers. The first assertion we did in Proposition 1.9, iii). For the second, we do as follows. Let $V \subset \mathbb{C}^n$, V open, $0 \in V$ and consider the diagram

$$\begin{array}{ccccc}
L & \xrightarrow{\ i\ } & V & \overset{\pi}{\underset{0}{\rightrightarrows}} & \mathbb{C}^k \\
\downarrow & & \downarrow & & \\
\mathbb{C}^{n-k} & \longrightarrow & \mathbb{C}^n & \overset{\pi}{\underset{0}{\rightrightarrows}} & \mathbb{C}^k
\end{array}$$

where π is the projection onto the last k coordinates, and the square in the left is a pullback. This pullback is transversal since $V \subset \mathbb{C}^n$ is open, and thus it is preserved by G. It follows then from Proposition 1.9, ii) that it is preserved by F, which since it preserves products, it also preserves the equalizer in the bottom row. It follows that F preserves the equalizer in the top row. Let now

$$W \xrightarrow{\ i\ } U \overset{h}{\underset{0}{\rightrightarrows}} \mathbb{C}^k ,$$

$U \subset \mathbb{C}^n$, $W \subset \mathbb{C}^{n-k}$, be an independent equalizer in $\mathcal{C}$. From the implicit function Theorem (Proposition 0.12) it follows that there is an open covering W_α of W, open sets $U_\alpha \subset U$, $W_\alpha = U_\alpha \cap W$, bi-holomorphic bijections $\phi_\alpha : V_\alpha \to U_\alpha$, $0 \in V_\alpha$ and $\Psi_\alpha : L_\alpha \to W_\alpha$, such that

$$\begin{array}{ccccc}
W & \xrightarrow{\ j\ } & U & \overset{h}{\underset{0}{\rightrightarrows}} & \mathbb{C}^k \\
\uparrow & & \uparrow & & \\
W_\alpha & \longrightarrow & U_\alpha & \overset{h}{\underset{0}{\rightrightarrows}} & \mathbb{C}^k \\
{\scriptstyle \Psi_\alpha}\uparrow & & \uparrow & & \\
L_\alpha & \longrightarrow & V_\alpha & \overset{\pi}{\underset{0}{\rightrightarrows}} & \mathbb{C}^k
\end{array}$$

commutes, where the bottom row is an equalizer of the previously considered form. Clearly F preserves the equalizer of the middle row. Let $U_0 \subset U$ be the complement of the set of zeros of h. There is an empty equalizer

$$\emptyset \longrightarrow U_0 \underset{0}{\overset{h}{\rightrightarrows}} \mathbf{C}^k$$

which, by Proposition 1.9, i), is also preserved by F. We have then

$$\begin{array}{ccccc} F(W) & \longrightarrow & F(U) & \underset{0}{\overset{F(h)}{\rightrightarrows}} & F^k \\ \uparrow & & \uparrow & & \\ F(W_\alpha) & \longrightarrow & F(U_\alpha) & \underset{0}{\overset{F(h)}{\rightrightarrows}} & F^k \end{array}$$

where we include also $\alpha = 0$. The bottom row is an equalizer in $\mathcal{E}ns$ for all α. By Proposition 1.9, iv) the vertical arrows are universal effective epimorphic families, while by 1.9, ii) each of the squares on the left is a is a pullback. It is easy then to check that the top row is also an equalizer. This finishes the proof.

We pass now to consider analytic rings in $\mathcal{E}ns$ furnished with a local morphism into C. Remark that C, as a functor $\mathcal{C} \to \mathcal{E}ns$, preserves open coverings.

1.11. DEFINITION. Let F be an object of $\mathcal{A}n(\mathcal{E}ns)$; F is a *local analytic ring* if there exists a local morphism $\pi : F \to \mathbf{C}$.

This definition means that for each open set U of $\mathbf{C}^n$, the square

$$\begin{array}{ccc} F(U) & \longrightarrow & F^n \\ {\scriptstyle \pi U}\downarrow & & \downarrow{\scriptstyle \pi^n} \\ U & \longrightarrow & \mathbf{C}^n \end{array}$$

is a pullback in $\mathcal{E}ns$. That is, the domain of definition of the U-ary operation of F is $F(U) = (\pi)^{-1}(U)$. If $s_1, \dots, s_n$ are elements of F, and $h : U \to \mathbf{C}$ is holomorphic, then the corresponding operation $\bar{h}$ is defined in $(s_1, \dots, s_n)$ iff $(\pi s_1, \dots, \pi s_n) \in U$; furthermore,

$$\pi \circ \bar{h}(s_1, \dots, s_n) = h(\pi s_1, \dots, \pi s_n).$$

(Compare with [3], Corollary 2 to Proposition 1.10.)

We will denote by $\mathcal{L}oc$ the full subcategory of $\mathcal{A}n(\mathcal{E}ns)$ whose objects are the local analytic rings.

1.12. OBSERVATION. Let us denote by C^* the open set $C^* = C - \{0\}$. Given any analytic ring F, all the elements of $F(C^*) \subset F(C) = F$ are invertible in the ring F, since the C^*-ary operation $1/x$ is defined in $F(C^*)$. When $\pi : F \to C$ is a local analytic ring, we have:

$$x \in F(C^*) \iff \pi(x) \in C^* \text{ for any given } x \text{ in } F.$$

Thus, x will be invertible in F provided that $\pi(x) \neq 0$. As it is well-known, this is equivalent to say that $F = F(C)$ is a local C-algebra with residual field C. In particular, π is unique since πC completely determines π.

Finally, we remark that for any analytic ring F, local or not, it follows easily (from the preservation of transversal pullbacks) that the set $F(C^*) \subset F$ is the set of units of the ring F.

1.13. PROPOSITION. *For each* $p \in C^n$, *we will denote with* $\mathcal{O}_{n,p}$ *the ring of germs of holomorphic functions in* p. *Then* $\mathcal{O}_{n,p}$ *is an analytic ring in* $\mathcal{E}ns_\Delta$. *We have* $\mathcal{O}_{n,p} = \operatorname{colim}_{p \in U} \mathcal{O}_n(U)$ *in* $\mathcal{A}n(\mathcal{E}ns_\Delta)$.

PROOF. Propositions 1.3 and 1.4.

1.14. OBSERVATION. For each open subset V of C^m, $\mathcal{O}_{n,p}(V)$ is the set of germs of holomorphic functions with values in V; by the Propositions 1.3 and 1.4:

$$\mathcal{O}_{n,p}(V) = (\operatorname{colim}_{p \in U} \mathcal{O}_n(U))(V) = \operatorname{colim}_{p \in U} \mathcal{O}_n(U)(V) = \operatorname{colim}_{p \in U} \mathcal{C}(U, V).$$

1.15. PROPOSITION. *There exists a morphism* $\pi : \mathcal{O}_{n,p} \to C$ *of analytic rings (necessarily unique) which makes* $\mathcal{O}_{n,p}$ *a local analytic ring. Then, it is clear that* π *is the unique morphism of C-algebras* $\mathcal{O}_{n,p} \to C$.

PROOF. $\pi = \operatorname{colim}(ev_p)$, where $ev_p : \mathcal{C}(U, V) \to C$ is the evaluation at the point p. By the Observation 1.14, the square

$$\begin{array}{ccc} \mathcal{O}_{n,p}(V) & \longrightarrow & \mathcal{O}_{n,p}(C)^m = (\mathcal{O}_{n,p})^m \\ {\scriptstyle \pi V} \downarrow & & \downarrow {\scriptstyle \pi^m} \\ V & \longrightarrow & C^n \end{array}$$

is a pullback, for any open set V in $\mathbb{C}^m$.

1.16. PROPOSITION. *($\mathcal{O}_{n,p}$ is the free local analytic ring in $n|p$-generators.) Let F be in $\mathfrak{Loc}$ and*

$$s_1, \dots, s_n \in F \quad \textit{such that} \quad \pi(s_i) = p_i \quad (1 \le i \le n),$$

where $\pi: F \to \mathbb{C}$ is the (unique) morphism of analytic rings from F to $\mathbb{C}$. Then there exists a unique morphism of analytic rings $\xi: \mathcal{O}_{n,p} \to F$ such that $\xi(z_i|p) = s_i$ $(1 \le i \le n)$ where z_i is the i-th projection $\mathbb{C}^n \to \mathbb{C}$.

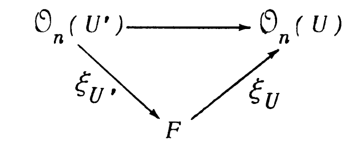
PROOF. Let U be open in $\mathbb{C}^n$ such that $p \in U$, $p = (p_1, \dots, p_n)$. Then

$$\pi(s_1, \dots, s_n) = (\pi s_1, \dots, \pi s_n) = p \in U.$$

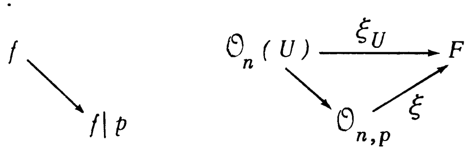
Thus $(s_1, \dots, s_n) \in F(U)$. By Proposition 2.8 there exists a unique morphism of analytic rings

$$\xi_U: \mathcal{O}_n(U) \to F \quad \text{such that} \quad \xi_U(z_i) = s_i \quad (1 \le i \le n).$$

If $U \subset U'$, clearly the diagram:



commutes (where the horizontal arrow is the restriction morphism). Therefore, by the universal property of the colimit, there exists a unique ξ : $\mathcal{O}_{n,p} \to F$ such that for each open U of $\mathbb{C}^n$ such that $p \in U$, the following diagram commutes:



1.17. PROPOSITION.

$$\mathcal{O}_{n,p} \otimes \mathcal{O}_{m,q} = \mathcal{O}_{n+m,(p,q)}$$

in $\mathfrak{An}(\mathfrak{Ens})$ and so also in $\mathfrak{Loc}$ (where $\otimes$ indicates the coproduct).

PROOF. It follows from Proposition 1.4 and the fact that products of opens form a base of open neighborhoods for the product topology.

1.18. THEOREM. *Let $\pi: G \to \mathbb{C}$ be a local analytic ring. Let I be an ideal* *of the* C*-algebra structure of* G. *Then, the following holds:*

i) There exists a unique local analytic ring, which we will denote G/I, *together with a unique morphism of analytic rings* $\nu : G \to G/I$, *such that* $G/I(\mathrm{C}) = G/I$ *and such that* ν_{C} *is equal to the canonical morphism into the quotient* $G \to G/I$ *as* C*-algebras.*

ii) If $\xi : G \to F$ *is a morphism of analytic rings such that* $\xi_{\mathrm{C}}(I) = \{0\}$ *then there exists a unique morphism of analytic rings* $\bar{\xi} : G/I \to F$ *such that the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\quad \xi \quad} & F \\ & {\scriptstyle \nu}\searrow \quad \nearrow{\scriptstyle \bar{\xi}} & \\ & G/I & \end{array}$$

PROOF. i) Let $\nu : G \to G/I$ be the quotient C-algebra, and $\pi : G/I \to \mathrm{C}$ be the unique morphism of C-algebras. We define $G/I(\mathrm{C}^n) = (G/I)^n$, and if U is any open subset of C^n, $U \subset \mathrm{C}^n$, we define $\nu U : G(U) \to G/I(U)$ to be the image of $G(U)$ by the map ν^n, as indicated in the following diagram:

$$\begin{array}{ccccc} G(U) & \xrightarrow{\nu U} & G/I(U) & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ G^n & \xrightarrow{\nu^n} & (G/I)^n & \longrightarrow & \mathrm{C}^n \end{array}$$

It follows that both squares are pullbacks since the maps in the lower row are surjective. We affirm there exists a unique structure of analytic ring on G/I which makes ν a (local) morphism of analytic rings. Effectively, for each open subset U of C^n and each holomorphic function $g : U \to \mathrm{C}$, we have to define $G/I(g)$ as indicated in the following diagram:

$$\begin{array}{ccc} G(U) & \xrightarrow{\quad G(g) \quad} & G \\ \downarrow & & \downarrow \\ G/I(U) & \xrightarrow{\quad G/I(g) \quad} & G/I \end{array}$$

Clearly it suffices to prove that if

$$(a_1, \ldots, a_n) \in G(U) \quad \text{and} \quad (b_1, \ldots, b_n) \in G(U)$$

are such that $a_i - b_i \in I \subset G$, then

$$G(g)(a_1, \dots, a_n) - G(g)(b_1, \dots, b_n) \in I.$$

We do as follows : $\pi(a_i) = \pi(b_i)$ since $a_i - b_i \in I$. Let

$$p = \pi(a_1, \dots, a_n) = \pi(b_1, \dots, b_n) \quad \text{and} \quad V = V_1 \times \dots \times V_n ,$$

$V_i \subset C$ be open, such that $p \in V \subset U$. From the fact that π is local it follows that

$$(a_1, \dots, a_n) \in G(V) \subset G(U) \quad \text{and} \quad (b_1, \dots, b_n) \in G(V) \subset G(U).$$

Take functions $h_i : V \times V \to C$ such that the equation

$$g(x_1, \dots, x_n) - g(z_1, \dots, z_n) = \sum_{i=1}^{n} (x_i - z_i) h_i(x_1, \dots, x_n, z_1, \dots, z_n)$$

holds in C (Corollary 0.10). Then, the following holds in G:

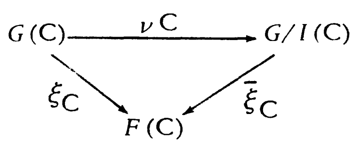
$$G(g)(a_1, \dots, a_n) - G(g)(b_1, \dots, b_n) = \sum_{i=1}^{n} (a_i - b_i) c_i ,$$

where $c_i = h_i(a_1, \dots, a_n, b_1, \dots, b_n)$. This shows that

$$G(g)(a_1, \dots, a_n) - G(g)(b_1, \dots, b_n) \in I.$$

It is immediate to check that in this way we determine a (finite) product preserving functor $G/I : \mathcal{C} \to \mathcal{E}ns$ together with a local natural transformation $\pi : G/I \to C$. Then, by the Theorem 1.10, G/I is an analytic ring.

ii) We know that with these hypotheses, there exists a unique morphism of C-algebras $\bar{\xi}_C : G/I(C) \to F(C)$ such that



commutes. This arrow induces the natural transformation $\bar{\xi}$ in an obvious way. The unicity comes from the fact that $\nu V : G(V) \to G/I(V)$ is surjective for each $V \in \mathcal{C}$, and so $\nu : G \to G/I$ is an epimorphism.

1.19. PROPOSITION. *If* $I \subset \mathcal{O}_{n,p}$ *and* $J \subset \mathcal{O}_{m,q}$ *are ideals, then*

$$\mathcal{O}_{n,p}/I \otimes \mathcal{O}_{m,q}/J = \mathcal{O}_{n+m,(p,q)}/(I, J)$$

in $\mathcal{A}n(\mathcal{E}ns)$ *and so in* $\mathcal{L}oc$, *where* (I, J) *is the ideal of* $\mathcal{O}_{n+m,(p,q)}$ *generated by the images of* I *and* J *by the canonical mappings :*

$$\mathcal{O}_{n,p} \longrightarrow \mathcal{O}_{n+m\,,\,(p\,,\,q)} \longleftarrow \mathcal{O}_{m,q}$$

PROOF. Proposition 1.17 and Theorem 1.18.

1.20. DEFINITION. An *analytic algebra in the Malgrange's sense* (Malgrange [7], page 32) is a C-algebra isomorphic to a quotient $\mathcal{O}_{n,p}/I$ with $I \subset \mathcal{O}_{n,p}$ any ideal. A *morphism of analytic algebras* is a morphism of C-algebras $u : A \to B$ such that there exists a germ h of holomorphic function $C^m \to C^n$ (around q) such that $h(q) = p$ and the following diagram commutes :

$$\begin{array}{ccc} \mathcal{O}_{n,p} & \xrightarrow{\quad h^* \quad} & \mathcal{O}_{m,q} \\ \downarrow & & \downarrow \\ A & \xrightarrow{\quad u \quad} & B \end{array}$$

where $h^*(f) = f \circ h$. We will denote this category $\mathfrak{Al}$.

1.21. PROPOSITION. *Let* $u : \mathcal{O}_{n,p} \to \mathcal{O}_{m,q}$ *be any map such that the diagram :*

$$\begin{array}{ccc} \mathcal{O}_{n,p} & \xrightarrow{\quad u \quad} & \mathcal{O}_{m,q} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & C & \end{array}$$

commutes. Then, u is a morphism of analytic algebras iff it is a morphism of analytic rings.

PROOF. If $u = h^*$, it is clear that u is a morphism of analytic rings. Now let u be a morphism of analytic rings. We take $h_i \mid q = u(z_i \mid p)$. We have a germ of holomorphic function $h : C^m \to C^n$ (around q)

$$h \mid q = (h_1 \mid q, \ldots, h_n \mid q)$$

such that

$$h(q) = (u(z_1 \mid p)(q), \ldots, u(z_n \mid p)(q)) =$$

$$= (z_1 \mid p(p), \ldots, z_n \mid p(p)) = p\,.$$

Clearly h defines a morphism of analytic rings $h^* : \mathcal{O}_{n,p} \to \mathcal{O}_{m,q}$; and we have

$$h^*(z_i \mid p) = h_i \mid q = u(z_i \mid p)\,.$$

From the Proposition 1.16 it follows that $h^* = u$. And so u is a morphism of analytic algebras.

1.22. PROPOSITION. *Let* $u : \mathcal{O}_{n,p} \to \mathcal{O}_{m,q}$ *be any map such that the diagram*

$$\begin{array}{ccc} \mathcal{O}_{n,p} & \xrightarrow{\;u\;} & \mathcal{O}_{m,q} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbb{C} & \end{array}$$

commutes. Then u *is a morphism of analytic algebras iff it is a morphism of* C*-algebras.*

PROOF. One of the implications is clear. We suppose that u is a morphism of C-algebras; as in the proof of the Proposition 1.21 we have a germ h such that $h^*(z_i \mid p) = u(z_i \mid p)$. Then, since u is a morphism of C-algebras, we have $h^*(l \mid p) = u(l \mid p)$ for any polynomial l. Let $f \mid p$ be any germ of analytic function, it is clear that for each power $\mathfrak{M}^k$ of the maximal ideal $\mathfrak{M} \in \mathcal{O}_{n,p}$ we have $f \mid p - l \mid p \in \mathfrak{M}^k$, where $l \mid p$ is the polynomial defined by a sufficiently long part of the Taylor's development of $f \mid p$. It follows that the polynomials are dense in the topology defined by the powers of the maximal ideal (Krull topology). This topology is Hausdorff, because it is clear that $\bigcap_{k \in \mathbb{N}} \mathfrak{M}^k = \{0\}$ (principle of analytic continuation, Cartan [2], Corollary 2, page 141). Since u and h^* are continuous (because both send the maximal ideal into the maximal ideal), we have that $u = h^*$. And so u is a morphism of analytic algebras.

1.23. PROPOSITION. *If* $I \subset \mathcal{O}_{n,p}$ *and* $J \subset \mathcal{O}_{m,q}$ *are ideals and*

$$u : \mathcal{O}_{n,p}/I \to \mathcal{O}_{m,q}/J$$

is any function such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{n,p}/I & \xrightarrow{\;u\;} & \mathcal{O}_{m,q}/J \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbb{C} & \end{array}$$

Then u *is a morphism of analytic rings (for the structures given in 1.18) iff* u *is a morphism of analytic algebras, iff* u *is a morphism of* C*-algebras.*

PROOF. Straightforward from 1.18, 1.21 and 1.22.

1.24. DEFINITION. A *Weil's C-algebra* (Weil [9], see also Dubuc [3], Definition 1.4) is a C-algebra X with the following properties:

i) it is local with maximal ideal I such that $X/I \approx C$,

ii) the dimension of X as C-vector space is finite,

iii) $I^{m+1} = 0$ for some natural number m.

The *height of* X is $h(X) = min\{m \in N \mid I^{m+1} = 0\}$.

We identify C with the subspace of the scalar multiples of 1 in X. If $x \in X$, then there exist: a unique scalar x_0 (the finite part of x) and unique nilpotent x_1 (the infinitesimal part of x) such that $x = x_0 + x_1$. So $X = C \oplus I$ (as C-vector spaces) and X has a canonical morphism $\pi_0 : X \to C$. If Z is any C-algebra, a morphism $\phi : Z \to X$ can be written in a unique way as $\phi = \phi_0 + \phi_1$, where $\phi_0 = \pi\phi$ is a morphism $Z \to C$ and ϕ_1 is a linear (but not multiplicative) map $Z \to C$. A morphism of Weil algebras is a morphism of C-algebras.

If $\xi_1, \dots, \xi_k$ are any elements of I which generate X, we can write $X = C[\xi_1, \dots, \xi_k]$, where the ξ_i satisfy a finite number of polynomial equations.

1.25. PROPOSITION. *All Weil's algebras are of the type* $\mathcal{O}_{n,p}/I$. *So any Weil's algebra has a unique structure of (local) analytic ring such that all morphism of C-algebras become morphisms of analytic rings.*

PROOF. Let X be a Weil's algebra, let I be the maximal ideal of X and let $\xi_1, \dots, \xi_n$ be elements of I such that $X = C[\xi_1, \dots, \xi_n]$. For each $z \in C^n$ and each $a \in N^n$ we write:

$$z^a = z_1^{a_1} \dots z_n^{a_n}, \quad a! = a_1! \dots a_n! \quad \text{and} \quad |a| = a_1 + \dots + a_n.$$

Let p be a point of C, $f|p \in \mathcal{O}_{n,p}$; and let $U \subset C$ and $f : U \to C$ holomorphic be a representative for $f|p$. Consider the Taylor's series development of f:

$$f(z) = \sum_{a \in N^n} \frac{D^a f(p)}{a!} (z-p)^a .$$

If $m = height(X)$, we can write

$$f(z) = \sum_{|\alpha| < m} \frac{D^{\alpha} f(p)}{\alpha!}(z-p)^{\alpha} + \sum_{|\alpha| > m} \frac{D^{\alpha} f(p)}{\alpha!}(z-p)^{\alpha}$$

Then the following mapping is well defined

$$\phi : \mathcal{O}_{n,p} \to X, \quad \phi(f|p) = \sum_{|\alpha| < m} \frac{D^{\alpha} f(p)}{\alpha!}(\xi_1, \dots, \xi_n)^{\alpha}.$$

It is clear that ϕ preserves 1 and is additive (because $D^{\alpha}(\cdot)(p)$ is additive). To check that ϕ is multiplicative is straightforward computation. Then ϕ is a morphism of C-algebras and it is clear that it is an epimorphism since all the polynomials are in $\mathcal{O}_{n,p}$. Finally, the rest of the statement follows from Propositions 1.18 and 1.23.

2. A-RINGED SPACES AND ANALYTIC SPACES.

We consider in this section analytic rings in the topos S_X of sheaves over a topological space X.

2.1. PROPOSITION. *Let X be any space, and let C_X be the sheaf of germs of continuous complex-valued functions defined in X. Then, C_X is an analytic ring in S_X and it preserves coverings. More explicitely:*

Let $C_X : \mathcal{C} \to S_X$ be the functor defined by:

$$\Gamma(H, C_X(U)) = Continuous\ (H, U),$$

for H open in X and $U \in \mathcal{C}$. Then

i) C_X preserves transversal pullbacks and the terminal object,

ii) C_X preserves open coverings.

PROOF. i) Let $U \to U_{\alpha}$ be a limit diagram in $\mathcal{C}$. Then, given any H open in X, the diagram

$$\text{Continuous}\ (H, U) \to \text{Continuous}\ (H, U_{\alpha})$$

is a limit diagram in $\mathcal{E}ns$. This means that $C_X(U) \to C_X(U_{\alpha})$ is a limit diagram in S_X. Thus, C_X preserves all limits, in particular i) holds.

ii) Let $U_{\alpha} \to U$ be an open covering in $\mathcal{C}$. Then, given any H open in X and $h : H \to U$ continuous, we have the diagrams

$$\begin{array}{ccc} h^{-1}(U_\alpha) & \longrightarrow & U_\alpha \\ \downarrow & & \downarrow \\ H & \xrightarrow{h} & U \end{array}$$

where $h^{-1}(U_\alpha) \hookrightarrow H$ is an open cover of H. This means that the family $C_X(U_\alpha) \rightarrowtail C_X(U)$ is an epimorphic family in S_X.

2.2. REMARK. Let C_X denote also the étale space over X whose fibers are the rings $C_{X,p}$ of germs of continuous complex-valued functions defined at p. Like in Propositions 1.13, 1.14 and 1.15, it follows easily that there is a local morphism $C_{X,p} \to C$, the value of the germ, which makes of $C_{X,p}$ a local analytic ring, for each p. All these morphisms collect together into a map $\pi : C_X \to C$. This map is continuous since C_X has the topology induced by its sections, and if σ is a section of C_X, $\pi\sigma$ is by definition a continuous complex-valued function. Since (the functor) C_X preserves products, the étale space of the sheaf $C_X(C^n)$ is the n-times iterated fibre product over X of $C_X \to X$, that we denote $C_X^n \to X$. Given any pair of open sets $H \subset X$, $U \subset C^n$, a section

$$\sigma \in \Gamma(H, C_X(U)) = \text{Continuous}(H, U)$$

is thus a n-tuple of sections $\sigma : H \to C_X^n$ such that $\pi^n \circ \sigma : H \to C^n$ factors through U. The étale space of the sheaf $C_X(U)$, that we also denote $C_X(U) \to X$, is the space

$$C_X(U) = (\pi^n)^{-1}(U) \subset C_X^n \to X.$$

Given any continuous function $f : X \to Y$, composing with f defines a morphism $f^* : C_{Y, f(p)} \to C_{X,p}$ of analytic rings in S_X, $f^* : f^*C_Y \to C_X$, where by f^*C_Y we indicate the inverse image in S_X of the sheaf C_Y in S_Y.

2.3. DEFINITION. An *A-ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X : \mathcal{C} \to S_X$ is an analytic ring in S_X furnished with a local morphism $l_X : \mathcal{O}_X \to C_X$ of analytic rings in S_X. Notice that from Theorem 1.10 and Proposition 1.8 it follows that the functor $\mathcal{O}_X$ preserves open covers. When the morphism l_X is injective, we will say that the A-ringed space is *reduced.*

2.4. REMARK. We will also denote $\mathcal{O}_X(\mathrm{C}) \to X$, or, according with our standard abuse of notation, simply $\mathcal{O}_X \to X$, the étale space of the sheaf $\mathcal{O}_X(\mathrm{C})$. The fibers $\mathcal{O}_{X,p}$ are analytic rings in $\mathcal{E}ns$ furnished with a morphism $l_{X,p}: \mathcal{O}_{X,p} \to \mathrm{C}_{X,p}$ into the ring of germs in p of continuous complex-valued functions. The fiber functor $p^*: S_X \to \mathcal{E}ns$ preserves finite limits, thus $l_{X,p}$ is a local morphism. It follows then since $\mathrm{C}_{X,p}$ is a local analytic ring that $\mathcal{O}_{X,p}$ is also a local analytic ring. All the morphisms $\mathcal{O}_{X,p} \to \mathrm{C}$ collect together into a map $\pi : \mathcal{O}_X \to \mathrm{C}$, that we denote also by π. This map is continuous since it is the composite

$$\mathcal{O}_X \xrightarrow{\ l_X\ } \mathrm{C}_X \xrightarrow{\ \pi\ } \mathrm{C},$$

where the second map is the one considered in Remark 2.2.

Since (the functor) $\mathcal{O}_X$ preserves products, the étale space of the sheaf $\mathcal{O}_X(\mathrm{C}^n)$ is the n-times iterated fiber product over X of $\mathcal{O}_X \to X$, that we denote $\mathcal{O}_X(\mathrm{C})^n \to X$, or, simply $\mathcal{O}_X^n \to X$. By definition of local morphism, given any pair of open sets $H \subset X$, $U \subset \mathrm{C}^n$, the following square is a pullback:

$$\begin{array}{ccc} \Gamma(H, \mathcal{O}_X(U)) & \rightarrowtail & \Gamma(H, \mathcal{O}_X(\mathrm{C}^n)) \\ {\scriptstyle l_X(U)} \downarrow & & \downarrow {\scriptstyle l_X(\mathrm{C}^n)} \\ \text{Continuous}\,(H, U) & \rightarrowtail & \text{Continuous}\,(H, \mathrm{C}^n) \end{array}$$

This means that a section $\sigma \in \Gamma(H, \mathcal{O}_X(U))$ can be identified with a n-tuple of sections $\sigma : H \to \mathcal{O}_X^n$ such that the n-tuple of their values in C belongs to U:

$$\begin{array}{ccc} H & \dashrightarrow & U \\ {\scriptstyle \sigma} \downarrow & & \downarrow \\ \mathcal{O}_X^n & \xrightarrow{\ \pi^n\ } & \mathrm{C}^n \end{array}$$

Thus the étale space of the sheaf $\mathcal{O}_X(U)$, that we also denote $\mathcal{O}_X(U) \to X$ is the space

$$\mathcal{O}_X(U) = (\pi^n)^{-1}(U) \subset \mathcal{O}_X \to X,$$

and σ can be identified with a continuous function (section)

$$H \xrightarrow{\sigma} (\pi^n)^{-1}(U) = \mathcal{O}_X(U).$$

We finally remark the the morphism l_X is unique. This is clear: Let $\phi, \phi' : \mathcal{O}_X \to C_X$ be any pair of morphisms. Then $\pi\phi = \pi\phi' = \pi$ since the fibers are local analytic rings in σ. Thus given any section σ of $\mathcal{O}_X$, $\phi(\sigma)$ and $\phi'(\sigma)$ are complex-valued functions with same values. Therefore, $\phi(\sigma) = \phi'(\sigma)$, which shows that $\phi = \phi'$.

NOTATION. Let $p \in X$ be a point of X, and let σ be a section of $\mathcal{O}_X$ defined in (a neighborhood of) p. We will denote by $\sigma | p$ the image of p by σ in $\mathcal{O}_{X,p}$ and by $\sigma(p)$ its value, that is, the complex number $\pi(\sigma | p)$.

2.5. DEFINITION. A *morphism* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *of one A-ringed space into another* is a pair

$$X \xrightarrow{f} Y, \quad f^*\mathcal{O}_Y \xrightarrow{\phi} \mathcal{O}_X$$

where f is a continuous function, $f^*\mathcal{O}_Y$ is the inverse image (in S_X) of the sheaf $\mathcal{O}_Y$ and ϕ is a morphism of analytic rings in S_X.

2.6. REMARK. The arrow ϕ can be interpreted as a family of morphisms of analytic rings in $\mathcal{E}_{n\mathcal{A}}$, $\phi_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ such that collected together they determine a continuous map between the corresponding étale spaces. Or, equivalently, such that given any section σ of $\mathcal{O}_Y$ defined in an open set $H \subset Y$, the composite $s|p = \phi_p(\sigma | f(p))$ defines a section of $\mathcal{O}_X$ in the open set $f^{-1}(H)$. Given any section defined in (a neighborhood of) $f(p)$, clearly

$$\phi_p(\sigma | f(p))(p) = \sigma(f(p)).$$

That is, ϕ always preserves the value of the sections. Furthermore, the diagrams

$$\begin{array}{ccccc} \mathcal{O}_{Y,f(p)} & \xrightarrow{\phi_p} & \mathcal{O}_{X,p} & & \\ \downarrow{\scriptstyle l_{Y,f(p)}} & & \downarrow{\scriptstyle l_{X,p}} & & \\ C_{Y,f(p)} & \longrightarrow & C_{X,p} & \longrightarrow & C_X \end{array}$$

commute since $\pi f^* l_{Y,f(p)} = \pi\, l_{X,p} \phi_p$, for all p, and, as before, sections of $C_{X,p}$ are determined by their values. Thus, ϕ always makes the following diagram commutative:

$$\begin{array}{ccc} f^*\mathcal{O}_Y & \xrightarrow{\ \phi\ } & \mathcal{O}_X \\ {\scriptstyle f^*(l_Y)}\downarrow & & \downarrow{\scriptstyle l_X} \\ f^*C_Y & \xrightarrow{\ f\ } & C_X \end{array}$$

2.7. PROPOSITION. *Let* $W \subset \mathbb{C}^n$ *be any open subset, and let* $\mathcal{O}_W$ *be the sheaf of germs of complex-valued holomorphic functions defined in* W. *Then* $\mathcal{O}_W$ *is an analytic ring in* S_W. *More explicitely,* $\mathcal{O}_W$ *is the functor*

$\mathcal{C} \to S_X$ *defined by* $\Gamma(V, \mathcal{O}_W(U)) = \mathcal{C}(V, U)$ *for* V *open in* W, $U \in \mathcal{C}$.

Furthermore, there is a morphism $l_W : \mathcal{O}_W \rightarrowtail C_W$ *defined by the inclusion* $\mathcal{C}(V, U) \subset$ *Continuous*(V, U) *which is local and determines a (reduced) A-ringed space* $(W, \mathcal{O}_W)$. *If* $p \in W$, *the fiber of* $\mathcal{O}_W$ *in* p *is* $\mathcal{O}_{W,p} = \mathcal{O}_{n,p}$.

PROOF. Exactly like in Proposition 2.1. Furthermore, it is clear that l_W is a local morphism.

The category of A-ringed spaces will be denoted $\mathcal{A}$. If $h : W \to V$ is a holomorphic function, composing with h, $h^* : \mathcal{O}_{V,h(p)} \to \mathcal{O}_{W,p}$ defines a morphism of analytic rings $h^* : h^*\mathcal{O}_V \to \mathcal{O}_W$ which, together with h itself, clearly determines a morphism of A-ringed spaces. In this way it is determined a full and faithful functor $i : \mathcal{C} \to \mathcal{A}$. We have:

2.8. THEOREM. *Given any A-ringed space* $(X, \mathcal{O}_X)$, *the following diagram is commutative:*

(1)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ i\ } & \mathcal{A} \\ {\scriptstyle \mathcal{O}_X}\downarrow & & \downarrow{\scriptstyle \mathcal{A}((X,\mathcal{O}_X),\,-)} \\ S_X & \xrightarrow{\ \Gamma\ } & \mathcal{E}ns \end{array}$$

That is, if U *is any open set* $U \subset \mathbb{C}^n$:

(2) $\quad \mathcal{A}((X, \mathcal{O}_X), (U, \mathcal{O}_U)) = \Gamma(X, \mathcal{O}_X(U)).$

PROOF. Let s be a section $s : X \to \mathcal{O}_X(U)$, that is, an n-tuple of sections $s = (s_1, \ldots, s_n)$, $s_i : X \to \mathcal{O}_X$, such that for all $p \in X$,

$$(s_1(p), \dots, s_n(p)) \in U.$$

Let (f, ϕ) be a morphism

$$f: X \to U, \quad \mathcal{O}_{U, f(p)} = \mathcal{O}_{n, f(p)} \xrightarrow{\phi_p} \mathcal{O}_{X, p}.$$

Then, the formulas $f(p) = (s_1(p), \dots, s_n(p))$, that is, $f = \pi s$, and

$$\phi_p(z_i \mid f(p)) = s_i \mid p,$$

establish the required natural bijection. Recall Proposition 1.16 which says that $\mathcal{O}_{n, f(p)}$ is a free local analytic ring in the generators $z_i \mid f(p)$.

2.9. COROLLARY. *The functor* $i : \mathcal{C} \to \mathcal{A}$ *is an analytic ring in* $\mathcal{A}$. *That is, it preserves terminal object and transversal pullbacks. Furthermore, it «represents» global sections. That is, we have*

$$\mathcal{A}((X, \mathcal{O}_X), i\mathbf{C}) = \Gamma(X, \mathcal{O}_X), \text{ where } i = i(\mathbf{C}) = (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$$

consists of the complex numbers structured with the sheaf of germs of holomorphic functions.

PROOF. In diagram (1) in the theorem, Γ preserves all limits and $\mathcal{O}_X$ is an analytic ring. For the second part, just specialize formula (2) in the theorem for $U = \mathbf{C}$.

We recall now some definitions of Grothendieck (Malgrange [8]) in order to show that the classical notion of analytic space gives an example of A-ringed spaces. The non-trivial part of this is the definition of the partial operations and then the preservation of pullbacks. However, all this follows easily from Theorems 1.10 and 1.18.

2.10. THEOREM. *Let* U *be an open subset of* $\mathbf{C}$ *and let* $\mathcal{Y}$ *be an arbitrary sheaf of ideals in* $\mathcal{O}_U$. *Let*

$$E = supp(\mathcal{O}_U/\mathcal{Y}), \quad E = \{p \in U \mid 1 \notin \mathcal{Y}|p \subset \mathcal{O}_{U,p}\})$$

or, equivalently,

$$E = \{ p \in U \mid h(p) = 0 \;\; \forall \; h|p \in \mathcal{Y}|p \}.$$

Then, the following holds:

i) There exists a unique analytic ring in S_E, *which we will denote*

$\mathcal{O}_E$, *and a local morphism of analytic rings* $l_E : \mathcal{O}_E \to C_E$, *such that*

$$\mathcal{O}_E(C) = i^*(\mathcal{O}_U/\mathcal{Y}) = \text{restriction of } (\mathcal{O}_U/\mathcal{Y}) \text{ to } E.$$

The pair $(E, \mathcal{O}_E)$ *is an A-ringed space furnished with a morphism of A-ringed spaces*

$$(E, \mathcal{O}_E) \xrightarrow{(i,q)} (U, \mathcal{O}_U),$$

where $i : E \hookrightarrow U$ *is the inclusion and* $q|p : \mathcal{O}_{U,p} \to \mathcal{O}_{E,p}$ *is the quotient*

$$\mathcal{O}_{U,p} = \mathcal{O}_{n,p} \to \mathcal{O}_{n,p}/\mathcal{Y}|p = \mathcal{O}_{E,p}$$

as constructed in Theorem 1.18.

ii) Let $(X, \mathcal{O}_X)$ *be any A-ringed space and let*

$$(f, \phi) : (X, \mathcal{O}_X) \longrightarrow (U, \mathcal{O}_U)$$

be such that

$$\phi_p(h|f(p)) = 0 \quad \text{for each} \quad p \in X \text{ and } h|f(p) \in \mathcal{Y}|f(p).$$

Then, there exists a unique morphism of A-ringed spaces

$$(f, \psi) : (X, \mathcal{O}_X) \to (E, \mathcal{O}_E)$$

such that the following diagram commutes:

$$\begin{array}{ccc} (X, \mathcal{O}_X) & & \\ {\scriptstyle (f,\psi)} \downarrow & \searrow^{(f,\phi)} & \\ (E, \mathcal{O}_E) & \xrightarrow{(i,q)} & (U, \mathcal{O}_U) \end{array}$$

PROOF. i) Consider

$$\begin{array}{ccc} \mathcal{O}_U & \xrightarrow{l_U} & i^*C_U \\ \downarrow & & \downarrow {\scriptstyle i^*} \\ \mathcal{O}_E = i^*(\mathcal{O}_U | Y) & \xrightarrow{l_E} & C_E \end{array}$$

in S_E. The factorization l_E indicated in the diagram follows by definition of E. We define $\mathcal{O}_E(C^k) = \mathcal{O}_E^k$, and for open $V \subset C^k$,

$$\mathcal{O}_E(V) = (l_E^k)^{-1} C_E(V).$$

Noticing that inverse limits are computed «fiberwise» in S_E, the proof follows from Theorem 1.18. Recall also Theorem 1.10.

ii) Since ϕ preserves the values of the sections (Remark 2.6), given $h \mid f(p) \in \mathcal{Y} \mid f(p)$, we have

$$h(f(p)) = \phi_p(h \mid f(p)) = 0(p) = 0.$$

Thus $f(p) \in E$. This shows that f factors through E. Working fibrewise, from Theorem 1.18 ii) it follows a factorization for ϕ_p :

$$\begin{array}{ccc} \mathcal{O}_{n,f(p)} & \xrightarrow{\ \phi_p\ } & \mathcal{O}_{X,p} \\ \downarrow & \swarrow_{\psi_p} & \\ \mathcal{O}_{E,f(p)} & & \end{array}$$

Given a section σ of $\mathcal{O}_E$ defined in a neighborhood F of $f(p)$ there is a neighborhood V of $f(p)$ in U and a section h of $\mathcal{O}_U$ defined in V such that

$$q(h \mid x) = \sigma \mid x \quad \text{for all } x \in F \cap V.$$

Then, the composite $s \mid x = \psi_x(\sigma \mid f(x))$ defined in a neighborhood Y of p in X, $f(Y) \subset F \cap V$, is equal to $s \mid x = \phi_x(h \mid f(x))$. Therefore it is a section of $\mathcal{O}_X$ defined in Y. This shows that the morphisms ψ_p collected together determine a morphism of A-ringed spaces.

2.11. DEFINITION. Let U be an open subset of C^n and let $\mathcal{Y}$ be a *coherent* sheaf of ideals in $\mathcal{O}_U$. The A-ringed space constructed in Proposition 2.10 is a *(local) model.* An *analytic space* is a ringed space (in C-algebras in the usual sense) $(X, \mathcal{O}_X)$ where every point $p \in X$ has an open neighborhood E such that the restriction of $(X, \mathcal{O}_X)$ to E is isomorphic to a model (defined above). The coherence of $\mathcal{Y}$ means that every point $p \in E \subset U$ of a model has an open neighborhood V in U such that $E \cap V$ is cut out of V by the vanishing of finitely many holomorphic functions defined in V. *A special model* is a model $(E, \mathcal{O}_E)$, $E \subset U$, cut out of U by the vanishing of the *same* finitely many holomorphic functions defined in U. Since the models are A-ringed spaces, it follows that any analytic space is an A-ringed space.

We will denote by $\mathfrak{D}$ and by $\mathfrak{L}$ the full subcategories of A-ringed spaces whose objects are the special models and the (local) models respectively. Notice that we have $i : \mathcal{C} \to \mathcal{D}$.

2.12. PROPOSITION. *Let* $U \subset \mathbb{C}^n$ *be open and* $h : U \to \mathbb{C}^k$ *holomorphic,* $h = (h_1, \dots, h_k)$, $h_i \in \Gamma(U, \mathcal{O}_U)$, *and let* $(E, \mathcal{O}_E)$ *be the special model defined by the pair* (U, h). *That is, defined by the sheaf of ideals* $\mathcal{Y}$ *in* $\mathcal{O}_U$, $\Gamma(V, \mathcal{Y}) = (h_1 | V, \dots, h_k | V)$ *for* V *open in* U. *Then the diagram*

$$(E, \mathcal{O}_E) \longrightarrow (U, \mathcal{O}_U) \underset{(0,\,0^*)}{\overset{(h,\,h^*)}{\rightrightarrows}} (\mathbb{C}^k, \mathcal{O}_{\mathbb{C}^k})$$

is an equalizer in $\mathcal{A}$:

PROOF. Let $(f, \phi) : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ be such that it equalizes (h, h^*) and 0. Then, for each $p \in X$, we have :

$$\phi_p(h_i | f(p)) = \phi_p h^*(z_i | hf(p)) = \phi_p\, 0^*(z_i | hf(p)) =$$
$$= \phi_p(0 | f(p)) = 0.$$

By Proposition 2.10, ii), this finishes the proof.

Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be special models defined respectively by pairs

$$(U, h),\ U \subset \mathbb{C}^n,\ U \xrightarrow{h} \mathbb{C}^k \quad \text{and} \quad (V, l),\ V \subset \mathbb{C}^m,\ V \xrightarrow{l} \mathbb{C}^r.$$

Then from Proposition 1.19 it follows easily that the special model defined by the pair

$$(U \times V, h \times l), \quad U \times V \xrightarrow{h \times l} \mathbb{C}^{k+r},$$

is a product of A-ringed spaces.

Let (f', ϕ) be a morphism between $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ given by a holomorphic function $f : U \to V$ in such a way that the following diagrams commute:

$$\begin{array}{ccc} E & \rightarrowtail & U \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ F & \rightarrowtail & V \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{O}_{E,p} & \longleftarrow & \mathcal{O}_{n,p} \\ {\scriptstyle \phi_p}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathcal{O}_{F,f(p)} & \longleftarrow & \mathcal{O}_{m,f(p)} \end{array}$$

(Notice that necessarily

$$(l_i f) | p \in (h_1 | p, \dots, h_k | p) \subset \mathcal{O}_n | p \quad \text{for all } i = 1, 2, \dots, r,\ p \in E\ ).$$

If (g', ψ) is another morphism given as above by a holomorphic function $g: U \to V$, then it follows in the same way that Proposition 2.12, that the special model defined by the pair

$$(U, (h, f \cdot g)), \quad U \xrightarrow{(h, f \cdot g)} \mathbb{C}^{k+m},$$

is an equalizer of A-ringed spaces between the pair of morphisms (f, ϕ) and (g, ψ). It follows from Proposition 2.12 and Theorem 2.8 that any morphism between special models is given *locally* in the above described way. Thus any pair of morphisms determines a coherent sheaf of ideals in $\mathcal{O}_U$ which defines a model which is their equalizer in the category of A-ringed spaces. It follows from these considerations that finite limits of special models exist and are models. Since to construct finite limits it suffices to do it locally, we see that the category $\mathcal{L}$ of models has finite limits. Corollary 2.9 says that the functor $i: \mathcal{C} \to \mathcal{L}$ is an analytic ring in $\mathcal{L}$. It is not difficult to check that in addition, it preserves open coverings.

3. THE THEORY OF ANALYTIC RINGS.

One could think that a natural candidate to be the theory of analytic rings would be the category $\mathcal{D}$. That is, that the functor $i: \mathcal{C} \to \mathcal{D}$ is the generic analytic ring. (As it happens in the algebraic case, where $\mathcal{D}$ = affine schemes; or in the C^∞-case, where $\mathcal{D}$ = affine C^∞-schemes.)

However $\mathcal{D}$ lacks finite limits, since in general the equalizer of a pair of morphisms in $\mathcal{D}$ is given by a coherent sheaf of ideals which is not determined by the global sections (because morphisms in $\mathcal{D}$ do not extend globally). The category $\mathcal{L}$ *does have* finite limits but the same lack of global sections does not allow a direct proof of the required universal property for the functor $i: \mathcal{C} \to \mathcal{L}$. One thinks then to construct a category like $\mathcal{D}$, but where morphisms *do* extend. For example the category $\mathcal{X}$ where objects are pairs (U, f), $U \subset \mathbb{C}^n$ open and $f: U \to \mathbb{C}^k$ holomorphic, and where morphisms $\phi: (U, f) \to (V, g)$ are holomorphic functions $h: U' \to V$ defined in an open neighborhood U' of the set of zeros of f, and such that

$$g_i h \in (f_i \mid U') \subset \mathcal{O}_n(U'),$$

two such functions being considered equal if their difference is in the ideal $(f_j | W)$ for a sufficiently small neighborhood W of the set of zeros of f. The problem here is then that the functor $i: \mathcal{C} \to \mathfrak{X}$ is *no longer an analytic ring*. Concretely, the morphisms that should be invertible in $\mathfrak{X}$ if i is to preserve transversal equalizers are only «locally invertible», and thus not invertible in $\mathfrak{X}$.

We will construct the theory of analytic rings in two steps. First, we construct a category with finite limits $\mathcal{F}$ (much like $\mathfrak{X}$) such that $i: \mathcal{C} \to \mathcal{F}$ preserves product and is universal with respect to product preserving functors. In a second step, by means of a calculus of fractions in $\mathcal{F}$, we obtain the theory of analytic rings.

Let $\mathcal{F}$ be the category whose objects are pairs (U, f), in which U is an open subset of $\mathbb{C}^n$ and $f: U \to \mathbb{C}^k$ is a holomorphic function. A morphism $h: (U, f) \to (V, g)$ is a holomorphic function $h: U \to V$ such that

$$g_i \circ h \in (f_1, \dots, f_k) \subset \mathcal{O}_n(U)$$

and two such functions $h, l: U \to V$ are considered equal if

$$h_i - l_i \in (f_1, \dots, f_k) \subset \mathcal{O}_n(U).$$

3.1. PROPOSITION. *The category $\mathcal{F}$ has finite limits.*

PROOF. i) $(\mathbb{C}^0, 0) = 1$.

ii) If $(U, f), (V, g) \in \mathcal{F}$, then $(U \times V, f \times g)$ is the product of (U, f), (V, g) in $\mathcal{F}$, where $f \times g: U \times V \to \mathbb{C}^k \times \mathbb{C}^r$ is the product map.

iii) Let (U, f) and (V, g) be objects of $\mathcal{F}$ and $h, l: (U, f) \to (V, g)$ two arrows in $\mathcal{F}$, then

$$(U, (f, h-l)) \xrightarrow{id_U} (U, f) \underset{l}{\overset{h}{\rightrightarrows}} (V, g)$$

is an equalizer diagram in $\mathcal{F}$. Suppose $s: (W, d) \to (U, f)$ equalizes h and l, it means that

$$(hs)_i - (ls)_i \in (h_1, \dots, h_k) \subset \mathcal{O}_n(W);$$

but $(hs)_i = h_i s$ and $(ls)_i = l_i s$ and so

$$(h_i - l_i) s = (h - l)_i s \in (h_1, \dots, h_k);$$

and for the others $f_i s \in (h_1, \dots, h_k)$ by hypothesis.

3.2. COROLLARY. *If* $(V, f) \in \mathcal{F}$ *then the diagram*

$$(V, f) \xrightarrow{id_V} (V, 0) \underset{0}{\overset{f}{\rightrightarrows}} (C^k, 0)$$

is an equalizer in $\mathcal{F}$.

PROOF. It is clear from the construction of equalizers in $\mathcal{F}$ (Proposition 3.1).

3.3. PROPOSITION. *We have a functor*

$$i : \mathcal{C} \to \mathcal{F} : U \mapsto (U, 0)$$

which is full and faithful and preserves finite products and terminal object; and i *has the following property: for any functor* $H : \mathcal{C} \to \mathcal{E}ns$ *which preserves finite products and terminal object, there exists a unique finite limits preserving functor* $\hat{H} : \mathcal{F} \to \mathcal{E}ns$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{F} \\ & {\scriptstyle H} \searrow \quad \swarrow {\scriptstyle \hat{H}} & \\ & \mathcal{E}ns & \end{array}$$

Furthermore, there exists an equivalence of categories between the functors $\mathcal{C} \to \mathcal{E}ns$ *that preserve finite products and terminal object and the functors* $\mathcal{F} \to \mathcal{E}ns$ *that preserve finite limits.*

PROOF. From the definition of the morphisms of $\mathcal{F}$ and the construction of products in $\mathcal{F}$ it is clear that i preserves finite products and terminal object. Now, let $H : \mathcal{C} \to \mathcal{E}ns$ be a functor which preserves finite products and terminal object. Notice that $H(C)$ is, in particular, a C-algebra. We define $\hat{H} : \mathcal{F} \to \mathcal{E}ns$ by the expression

$$\hat{H}(U, f) = Nat[\mathcal{F}[(U, f), i(\cdot)], H]$$

then

$$\hat{H}(i(U)) = Nat[\mathcal{F}[i(U), i(\cdot)], H] = Nat[\mathcal{C}[U, \cdot], H] = H(U),$$

because i is full and faithful and Yoneda's lemma. We must see that $\hat{H}$ preserves finite limits. We will only prove in detail that the diagram

$$\hat{H}(U,f) \longrightarrow H(U) \underset{0}{\overset{H(f)}{\rightrightarrows}} H(\mathbb{C}^k)$$

is an equalizer in $\mathcal{E}n\Delta$, where the left arrow is the map $\xi \mapsto \xi U(id_U)$. It is then straightforward to show that $\hat{H}$ preserves all equalizers, finite products and terminal object. Let ξ be an element of $\hat{H}(U,f)$ and $g: U \to V$ holomorphic. We consider the diagram

$$\begin{array}{ccc}
\mathcal{F}[(U,f), iV] & \xrightarrow{\quad \xi V \quad} & H(V) \\
\uparrow & (1) & \uparrow H(g) \\
\mathcal{F}[(U,f), iU] & \xrightarrow{\quad \xi U \quad} & H(U) \\
\downarrow & (2) & \downarrow H(f_i) \\
\mathcal{F}[(U,f), i\mathbb{C}] & \xrightarrow{\quad \xi \mathbb{C} \quad} & H(\mathbb{C})
\end{array}$$

where the vertical arrows on the left are compositions with g in (1) and with f_i in (2). Since $f_i = 0$ in $\mathcal{F}[(U,f), i\mathbb{C}]$, from the diagram (2), it follows that

$$H(f_i)(\xi U(id_U)) = 0 \quad \text{for all } i.$$

On the other hand, if $p \in H(U)$ is such that $H(f_i)(p) = 0$ for all i, the expression $\xi V(g) = H(g)(p)$ defines a unique natural transformation ξ such that $\xi U(id_U) = p$; because, if $g = g'$ in $\mathcal{F}[(U,f), i(V)]$, by definition

$$(g_k - g'_k) = (g - g')_k = (g - g')_k \in (f_i))$$

then $H(g_k) - H(g'_k) = 0$ (for all k). This shows that the definition of of $\xi V(g)$ does not depend of the choice of g. And the diagram (1) shows that ξ as defined above is the unique one such that $\xi U(id_U) = 0$.

3.4. PROPOSITION. *The functor* $i: \mathcal{C} \to \mathcal{F}$ *is universal for product preserving functors* $H: \mathcal{C} \to \mathcal{E}$ *into any category* $\mathcal{E}$ *with finite limits.*

PROOF. It is the same that the one we will give in Proposition 3.6.

3.5. PROPOSITION. *Let* Σ *be the subset of* $Fl(\mathcal{F})$ *defined by:*

$\sigma \in \Sigma \iff \hat{H}(\sigma)$ *is invertible for all analytic rings* H *in* $\mathcal{E}n\Delta$.

Let $\mathcal{F}[\Sigma^{-1}] = \mathfrak{A}n$, *and* $l: \mathcal{C} \to \mathfrak{A}n$ *be the composite*

$$\mathcal{C} \xrightarrow{\ i\ } \mathcal{F} \xrightarrow{\ p_\Sigma\ } \mathcal{A}_n .$$

Then the following holds:

i) $\mathcal{A}_n$ *has finite limits (and* p_Σ *preserves finite limits), and* $l : \mathcal{C} \to \mathcal{A}_n$ *is an analytic ring in* $\mathcal{A}_n$.

ii) Given any analytic ring H *in* $\mathcal{E}_{n\Delta}$*, there exists a unique finite limits preserving functor* $\bar{H} : \mathcal{A}_n \to \mathcal{E}_{n\Delta}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ l\ } & \mathcal{A}_n \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle \bar{H}} & \\ & \mathcal{E}_{n\Delta} & \end{array}$$

Furthermore, there exists an equivalence of categories between the analytic rings $\mathcal{C} \to \mathcal{E}_{n\Delta}$ *in* $\mathcal{E}_{n\Delta}$*, and the finite limits preserving functors* $\mathcal{A}_n \to \mathcal{E}_{n\Delta}$*. In symbols:* $\mathcal{A}_n(\mathcal{E}_{n\Delta}) \simeq \mathcal{E}_{n\Delta}^{(\mathcal{A}_n)}$.

PROOF. Let $H : \mathcal{C} \to \mathcal{E}_{n\Delta}$ be an analytic ring. Then we have:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\ i\ } & \mathcal{F} & \xrightarrow{\ p_\Sigma\ } & \mathcal{A}_n \\ & {\scriptstyle H}\searrow & \downarrow{\scriptstyle \hat{H}} & \swarrow{\scriptstyle \bar{H}} & \\ & & \mathcal{E}_{n\Delta} & & \end{array}$$

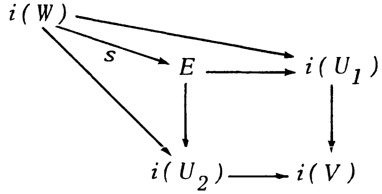
By Propositions 3.3 and 0.8 it follows ii). Also we see that $\mathcal{A}_n$ has finite limits (and p_Σ preserves finite limits). It remains to see that l is an analytic ring. Let

$$\begin{array}{ccc} W & \longrightarrow & U_1 \\ \downarrow & & \downarrow{\scriptstyle f} \\ U_2 & \xrightarrow{\ g\ } & V \end{array}$$

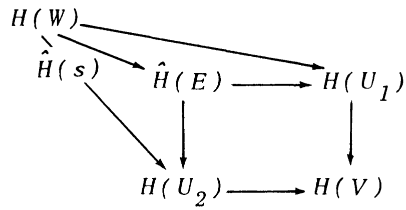
be a transversal pullback in $\mathcal{C}$. Since $\mathcal{F}$ has finite limits, there exists $E \in \mathcal{F}$ such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & i(U_1) \\ \downarrow & & \downarrow{\scriptstyle if} \\ i(U_2) & \xrightarrow{\ ig\ } & i(V) \end{array}$$

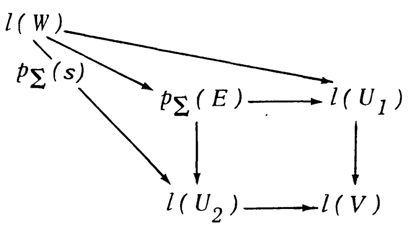
is a pullback in $\mathcal{F}$. And since i is a functor there exists a unique s: $i(W) \to E$ such that the diagram

(1) 

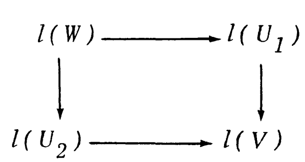
is commutative. For any $H \in \mathcal{A}n(\mathcal{E}_{n\Delta})$, $\hat{H}$ sends the diagram (1) into the following commutative diagram



But $\hat{H}$ preserves finite limits and H preserves transversal pullbacks; then $H(W)$ and $\hat{H}(E)$ are pullbacks in $\mathcal{E}_{n\Delta}$; so $\hat{H}(s)$ is an isomorphism and so $s \in \Sigma$, then $p_\Sigma(s)$ is an isomorphism and p_Σ sends the diagram (1) into the commutative diagram



But p_Σ preserves finite limits, thus $p_\Sigma(E)$ is a pullback; and since $p_\Sigma(s)$ is an isomorphism the following diagram is a pullback:



A similar argument shows that l preserves the terminal object.

3.6. PROPOSITION *(Completeness Theorem). Let $H: \mathcal{C} \to \mathcal{E}$ be an analytic ring in any category $\mathcal{E}$ with finite limits. Then there exists a unique finite* *limits preserving functor* $H: \mathfrak{A}n \to \mathcal{E}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;l\;} & \mathfrak{A}n \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle \bar{H}} & \\ & \mathcal{E}. & \end{array}$$

Furthermore, this establishes an equivalence of categories between $\mathfrak{A}n(\mathcal{E})$ *and* $\mathcal{E}(\mathfrak{A}n)$, *the category of finite limits preserving functors* $\mathfrak{A}n \to \mathcal{E}$.

PROOF. The proposition is true for $\mathcal{E} = \mathcal{E}ns$. Let $\mathfrak{M}$ be any category, we will prove it for $\mathcal{E} = \mathcal{E}ns^{\mathfrak{M}}$. Let $H \in \mathfrak{A}n(\mathcal{E}ns^{\mathfrak{M}})$. Since $\mathcal{E}ns^{\mathfrak{M}}$ is an exponential in $\mathcal{C}at$ (the category of categories), we have by adjunction a functor $H': \mathfrak{M} \to \mathcal{E}ns^{\mathcal{C}}$ such that $H'(M)(U) = H(U)(M)$. For any $M \in \mathfrak{M}$, $H'(M) \in \mathfrak{A}n(\mathcal{E}ns)$. Then, we have a functor $H': \mathfrak{M} \to \mathcal{E}ns^{(\mathfrak{A}n)}$. But $\mathcal{E}ns^{(\mathfrak{A}n)}$ is a full subcategory of $\mathcal{E}ns^{\mathfrak{A}n}$, so again by adjunction there is a functor

$$\bar{H}: \mathfrak{A}n \to \mathcal{E}ns^{\mathfrak{M}} \quad \text{such that} \quad \bar{H}(E)(M) = H'(M)(E).$$

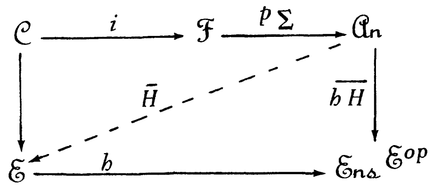
The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;l\;} & \mathfrak{A}n \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle \bar{H}} & \\ & \mathcal{E}ns^{\mathfrak{M}} & \end{array}$$

clearly commutes. Finally, let $\mathcal{E}$ be any category with finite limits and $H \in \mathfrak{A}n(\mathcal{E})$. Composing with Yoneda's functor we have:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;l\;} & \mathfrak{A}n \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \overline{hH}} \\ \mathcal{E} & \xrightarrow{\;h\;} & \mathcal{E}ns^{\mathcal{E}^{op}} \end{array}$$

Consider now the category $\mathfrak{F}$ of Proposition 3.1. The fact that the Yoneda functor h is full and faithful, together with Corollary 3.2 and the existence of equalizers in $\mathcal{E}$, implies that there is a factorization $\mathfrak{F} \to \mathcal{E}$ as indicated in the diagram below. The factorization $\mathfrak{A}n \to \mathcal{E}$ follows then by the universal property of $p_{\Sigma}: \mathfrak{F} \to \mathfrak{A}n$.



This finishes the proof.

This shows that $l: \mathcal{C} \to \mathcal{A}_n$ is the generic analytic ring.

3.7. REMARK. It follows from the results of Section 2 that there is a finite limits preserving functor $\mathcal{A}_n \to \mathcal{L} \subset \mathcal{A}$. This is the functor *Spec* ; we do not know if this functor if full and faithful as it is in the corresponding algebraic and C^{∞}-situtations. The consideration of the model of Synthetic Differential Geometry determined by the site $\mathcal{A}_n$ may be useful to the study of the model determined by the site $\mathcal{L}$. That is, to the theory of Analytic Spaces. It would be interesting to show that $\mathcal{L} = \mathcal{A}_n$.

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Facultad de Ciencias Exactas
Departamento de Matematicas
Ciudad Universitaria, Pabellon 1
1428 - BUENOS-AIRES
ARGENTINE