PARAMETERIZING AND FITTING BOUNDED ALGEBRAIC CURVES AND SURFACES

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Abstract

This paper introduces a new approach to implicit algebraic curve and surface fitting. Two families of polynomials with bounded zero sets are presented. Members of these families have the same number of degrees of freedom as general polynomials of the same degree. Methods for fitting members of these families of polynomials to measured data points are described. Experimental results of fitting curves to sets of points in $\mathcal{R}^2$ and surfaces to sets of points in $\mathcal{R}^3$ are presented.

1 Introduction

In the last few years several researchers have started using algebraic curves and surfaces of high degree as geometric models or shape descriptors, in different model-based computer vision tasks. Typically, the input for these tasks is either an intensity image or dense range data. While early approaches to model-based computer vision focussed on polyhedral objects or the blocks world, more recent work has considered curved objects which are typically modelled by collections of curved primitives. In nearly all of this work, the primitives have been natural quadrics (spheres, ellipsoids, cylinders, and cones), superquadrics, or a well-defined subset of generalized cylinders. Algebraic approaches for recognizing and locating objects represented by implicit (or parametric) polynomial surfaces of degree higher than 2 have only recently been proposed.

One of the fundamental problems in building a recognition and positioning system based on implicit curves and surfaces is how to fit these curves and surfaces to data. This process is necessary for automatically constructing object models from range or intensity data and for building intermediate representations from observations during recognition. Several methods are available for extracting straight line segments, planar patches, quadratic arcs, and quadric surface patches from 2D edge maps and 3D range images. Recently, methods have also been developed for fitting algebraic curve and surface patches of arbitrary degree. Detailed surveys of previous work on the subject can be found in [2, 5, 6, 7]. This paper primarily addresses the problem of fitting bounded algebraic curves and surfaces to point data.

Although implicit algebraic curves and surfaces have many good properties which make them the natural choice for object recognition and positioning, parametric curves and surfaces still outperform them in a fundamental area. More stable or robust algorithms are known to approximate sets of measured data points by parametric curves and surfaces than by their implicit counterparts. One of the main problems is that, while the data sets are always bounded, the algebraic curves or surfaces fitted to them are, in most cases, unbounded. Additionally, very small changes in the coefficients of the polynomials often produce very large changes in the global shape of the curve or surface, and the notion of locality is generally lost during the fitting process. One possible solution to this problem is to constrain the coefficients of the defining polynomials in such a way that the curves or surfaces defined by these polynomials are always bounded. In general, there are two ways to introduce these constraints: first, constraints on the coefficients can be expressed implicitly by a system of equality and inequality equations. Alternatively, the coefficients of the polynomials can be specified as functions of unbounded parameters; consequently the polynomials are restricted to a family with the desired properties. In this paper, we concentrate on two particular families of parameterized polynomials that leads to certain computational advantages during fitting. A well known example of a parameterized family of bounded implicit surfaces is the family of superquadrics (actually, superellipsoids [1]) which have up to eight shape parameters if bending and tapering are considered (plus six more for pose). However, eight degrees of freedom leaves little shape control, and something more general is needed!

As an example of the presented parameterizations, consider the case of conics, algebraic curves defined by

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second degree polynomials in two variables. Ellipses and circles are the only bounded conics. A second degree polynomial in two variables can be written in matrix form in two different ways

1. $f(x_1, x_2) = X^t AX + g(x_1, x_2)$
2. $f(x_1, x_2) = X^t AX$

where in the first case A is a symmetric 2×2 matrix, $X = (x_1, x_2)^t$, and $g(x_1, x_2)$ is a polynomial of degree < 2, and in the second case A is a symmetric 3×3 matrix, and $X = (1, x_1, x_2)^t$. With the first formulation, if the polynomial $f(x_1, x_2) = X^t AX + g(x_1, x_2)$ defines an ellipse, the matrix A is positive definite. With the second formulation, every ellipse can be represented as a level set of the polynomial $f(x_1, x_2) = X^t AX$ where A is positive definite. A matrix A is positive definite if and only if $A = BB^t$ for certain nonsingular matrices B which can also be taken symmetric. Thus, every ellipse can be represented as either the set of zeros of an element of the family of quadratic polynomials $\{f = X^t B^2 X + g : |B| \neq 0\}$, with $X = (x_1, x_2)^t$ and $B \in \mathcal{R}^{2\times 2}$ symmetric, or as a level set of an element of the family of quadratic polynomials $\{f = X^t B^2 X : |B| \neq 0\}$, with $X = (1, x_1, x_2)^t$, and $B \in \mathcal{R}^{3\times 3}$ symmetric. In the first case, the coefficients of second degree of f are quadratic functions of the parameters (the elements of the matrix B), and the other coefficients are linear functions of the parameters (the coefficients of g). In the second case, the coefficients of the members of this family are quadratic functions in the elements of the matrix B.

In this paper we generalize the previous constructions to higher degree polynomials. Not every bounded curve or surface of degree d can be represented as a level set of a member of one of these families, but we show that these families are rich enough in terms of shape description power. In particular, both have as many degrees of freedom as a general polynomial of the same degree.

Finally, let us mention that Keren, Cooper, and Subrahmonia [3] have independently developed methods similar to us for the limited case of quartic curves and surfaces.

2 Algebraic curve and surface fitting

In most work in computer aided design and computer vision, surfaces are represented parametricly as a smooth vector function $s : \mathcal{R}^2 \to \mathcal{R}^3$ where each coordinate of s is typically either a polynomial or ratio of polynomials.

An implicit surface is the set of zeros of a smooth function $f : \mathcal{R}^3 \to \mathcal{R}$ of three variables:

$$Z(f) \;=\; \{(x_1, x_2, x_3) : f(x_1, x_2, x_3) = 0\} \,.$$

Similarly, an implicit 2D curve is the set $Z(f) = \{(x_1, x_2) : f(x_1, x_2) = 0\}$ of zeros of a smooth function $f : \mathcal{R}^2 \to \mathcal{R}$ of two variables. The curves or surfaces are *algebraic* if the functions are polynomials.

Other common surface representations such as the quadric surfaces (e.g. cones, ellipsoids, hyperboloids, etc.) admit both a parametric and implicit form. However, there are some algebraic surfaces (third order and higher) that can only be represented implicitly.

We now consider the process of fitting an algebraic curve or surface to measured data points. The first step is to restrict the functions which define the curves or surfaces to a family parameterized by a finite number of parameters. Let $\phi : \mathcal{R}^{r+n} \to \mathcal{R}$ be a smooth function, and for certain $u = (u_1, \ldots, u_r)^t$, let us consider the maps $f : \mathcal{R}^n \to \mathcal{R}$ which can be written as

$$f(x) \;=\; \phi_u(x) \;\equiv\; \phi(u, x) \,.$$

We refer to $u_1, \ldots, u_r$ as the *parameters* and to $x_1, \ldots, x_n$ as the *variables*. We say that ϕ is the *parameterization* of the family $\mathcal{F} = \{f : \exists u \; f = \phi_u\}$. The set of zeros $Z(f)$ of a member f of $\mathcal{F}$ is a 2D curve when $n = 2$, and a surface when $n = 3$. As an example, consider the family of polynomials given by:

$$\phi(u, x) \;=\; x_1^2 + x_2^2 - u^2 .$$

In this case, there is one parameter u and two variables x_1, x_2; clearly, the zero set of a member of this family is simply a circle centered at the origin, and the one shape parameter determines its radius.

Given a finite set of n-dimensional ($n = 2$ or $n = 3$) data points $\mathcal{D} = \{p_1, \ldots, p_q\}$, the problem of fitting an implicit curve or surface $Z(f)$ to the data set $\mathcal{D}$ corresponds to determining the $\hat{f} \in \mathcal{F}$ that minimizes the mean square distance

$$\frac{1}{q} \sum_{i=1}^{q} \mathrm{dist}(p_i, Z(f))^2 \tag{1}$$

from the data points to the curve or surface $Z(f)$.

Unfortunately, there is no closed form expression for the distance from a point to a generic implicit curve or surface, not even for algebraic curves or surfaces, and iterative methods are required to compute it. This makes the minimization of (1) computationally impractical. Thus, we seek approximations to the distance function which are computationally practical.

Kriegman and Ponce [2, 5] have shown that elimination theory can be used to construct a closed-form expression for the *exact distance* from a point $x \in \mathcal{R}^n$ to the zero set $Z(f)$ of a polynomial. The distance δ is given by the following system of equations:

$$\left\{ \begin{array}{rcl} f(\hat{y}) & = & 0, \\ \delta^2 - \|x - \hat{y}\|^2 & = & 0, \\ (x - \hat{y}) - \lambda \nabla f(\hat{y}) & = & 0. \end{array} \right. \tag{2}$$

The variables $\hat{y}_1, \ldots, \hat{y}_n, \lambda$ are eliminated among these $n+2$ equations, yielding a new equation

$$D(\delta, x, f) \;=\; 0 \,, \tag{3}$$

where D is a polynomial in the distance δ, with coefficients that are polynomial in the coordinates $x_1, \ldots, x_n$, and the coefficients of the polynomial f. For a given polynomial f, the distance δ is the minimum positive root of this polynomial, and it can be found by some numerical root-finding algorithm.

Taubin [6, 7] has shown that another alternative to approximately solve the original computational problem, i.e., the minimization of (1), is to replace the real distance from a point to an implicit curve or surface by the first order approximation,

$$\text{dist}(x, Z(f))^2 \approx \frac{f(x)^2}{\|\nabla f(x)\|^2} . \tag{4}$$

The mean value of this function on a fixed set of data points

$$\Delta^2(u) = \frac{1}{q} \sum_{i=1}^{q} \frac{f(p_i)^2}{\|\nabla f(p_i)\|^2} . \tag{5}$$

is a smooth nonlinear function of the parameters, and can be *locally* minimized using well established nonlinear least squares techniques.

3 Euler's theorem

In this section we state the version of Euler's theorem which shows how to write a polynomial of even degree d as a quadratic form $X^t \Psi X$ in the monomials of degree $d/2$.

From now on, polynomials will be written expanded in Taylor series at the origin

$$f(x) = f(x_1, \ldots, x_n) = \sum_{\alpha} \frac{1}{\alpha!} F_\alpha \, x^\alpha , \tag{6}$$

where the vector of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)^t$ is a *multiindex* of *size* $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ is a multiindex factorial, $F_\alpha \in \mathcal{R}$ is a coefficient of degree $|\alpha|$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a *monomial* of degree $|\alpha|$. The coefficients of f are equal to the partial derivatives of order d evaluated at the origin, and only finitely many coefficients are different from zero. A polynomial is homogeneous, or a *form*, if all its terms are of the same degree

$$\psi(x) = \sum_{|\alpha|=d} \frac{1}{\alpha!} \Psi_\alpha \, x^\alpha . \tag{7}$$

Equivalently, a polynomial $\psi(x)$ is a form of degree d if and only if $\psi(\theta x) \equiv \theta^d \psi(x)$ is a polynomial identity in $n+1$ variables $\theta, x_1, \ldots, x_n$.

Forms are related to non-homogeneous polynomials in two ways. We will use both methods to define different parameterizations of families of polynomials with bounded level sets in section 5. In the first place, every polynomial $f(x)$ of degree d can be written in a unique way as a sum of forms

$$f(x) = \sum_{k=0}^{d} f_k(x) , \tag{8}$$

where $f_k(x)$ is a form of degree k, and $f_d(x) \neq 0$. In the second place, by introducing homogeneous coordinates, every curve or surface described in Euclidean space by a polynomial in n variables, can be described in projective space by an associated form in $n+1$ variables. If $\psi(x_0, \ldots, x_n)$ is a form of degree d in $n+1$ variables, and $f(v_1, \ldots, v_n)$ is a (non-homogeneous) polynomial of degree $\leq d$ in n variables, the one-to-one correspondence is given by

$$\begin{aligned} \psi(x_0, \ldots, x_n) &= x_0^d \, f(x_1/x_0, \ldots, x_n/x_0) \\ f(v_1, \ldots, v_n) &= \psi(1, v_1, \ldots, v_n) . \end{aligned} \tag{9}$$

In other words, every polynomial in n variables is the restriction of a form in $n+1$ variables to the hyperplane $\{x : x_0 = 1\}$, and every form in $n+1$ variables is totally determined by its restriction to this hyperplane.

The set of monomials $\{x^\alpha/\sqrt{\alpha!} : |\alpha| = d\}$ of degree d lexicographically ordered, define a vector of dimension h_d, which we will denote $X_{[d]}(x)$. For example,

$$X_{[3]}(x_1, x_2) = \left(\tfrac{1}{\sqrt{6}} x_1^3 \quad \tfrac{1}{\sqrt{2}} x_1^2 x_2 \quad \tfrac{1}{\sqrt{2}} x_1 x_2^2 \quad \tfrac{1}{\sqrt{6}} x_2^3 \right)^t .$$

Also, for every pair of integers, (j, k) such that $j + k = d$, the set of coefficients

$$\left\{ \sqrt{\tfrac{1}{\alpha! \, \beta!}} \, \Psi_{\alpha+\beta} \; : \; |\alpha| = j, |\beta| = k \right\}$$

lexicographically ordered in both indices, defines an $h_j \times h_k$ matrix which we will denote $\Psi_{[j,k]}(x)$. For example, for a fourth degree form in three variables we have

$$\Psi_{[2,2]} = \begin{pmatrix}
\frac{\Psi_{(400)}}{2} & \frac{\Psi_{(310)}}{\sqrt{2}} & \frac{\Psi_{(301)}}{\sqrt{2}} & \frac{\Psi_{(220)}}{2} & \frac{\Psi_{(211)}}{\sqrt{2}} & \frac{\Psi_{(202)}}{2} \\
\frac{\Psi_{(310)}}{\sqrt{2}} & \frac{\Psi_{(220)}}{1} & \frac{\Psi_{(211)}}{1} & \frac{\Psi_{(130)}}{\sqrt{2}} & \frac{\Psi_{(121)}}{1} & \frac{\Psi_{(112)}}{\sqrt{2}} \\
\frac{\Psi_{(301)}}{\sqrt{2}} & \frac{\Psi_{(211)}}{1} & \frac{\Psi_{(202)}}{1} & \frac{\Psi_{(121)}}{\sqrt{2}} & \frac{\Psi_{(112)}}{1} & \frac{\Psi_{(103)}}{\sqrt{2}} \\
\frac{\Psi_{(220)}}{2} & \frac{\Psi_{(130)}}{\sqrt{2}} & \frac{\Psi_{(121)}}{\sqrt{2}} & \frac{\Psi_{(040)}}{2} & \frac{\Psi_{(031)}}{\sqrt{2}} & \frac{\Psi_{(022)}}{2} \\
\frac{\Psi_{(211)}}{\sqrt{2}} & \frac{\Psi_{(121)}}{1} & \frac{\Psi_{(112)}}{1} & \frac{\Psi_{(031)}}{\sqrt{2}} & \frac{\Psi_{(022)}}{1} & \frac{\Psi_{(013)}}{\sqrt{2}} \\
\frac{\Psi_{(202)}}{2} & \frac{\Psi_{(112)}}{\sqrt{2}} & \frac{\Psi_{(103)}}{\sqrt{2}} & \frac{\Psi_{(022)}}{2} & \frac{\Psi_{(013)}}{\sqrt{2}} & \frac{\Psi_{(004)}}{2}
\end{pmatrix}$$

Now we have all the necessary elements to state Euler's theorem in the form we need. The classical proof for $j = 1$ can be found in [10]. For the general case, which follows by induction, see [8].

Lemma 1 (Euler's theorem) *For every form ψ of degree $d = j+k$, we have*

$$\begin{aligned} \binom{d}{j} \psi(x) &= X_{[j]}^t(x) \Psi_{[j,k]} X_{[k]}(x) \\ &= \sum_{|\alpha|=j} \sum_{|\beta|=k} \frac{1}{\alpha! \beta!} \Psi_{\alpha+\beta} \, x^{\alpha+\beta} . \end{aligned}$$

In particular, every form ψ of even degree $d = 2k$ can be written as a quadratic form in the monomials of degree $k = d/2$.

4 A quadratic parameterization of even degree forms

For every form ψ of even degree $d = 2k$, let

$$\begin{aligned} Q(\psi)(x) &= X_{[k]}^t(x)\,\Psi_{[k,k]}^2\,X_{[k]}(x) \\ &= \sum_{|\alpha|=k}\sum_{|\beta|=k}\sum_{|\gamma|=k} \tfrac{1}{\alpha!\beta!\gamma!}\,\Psi_{\alpha+\beta}\,\Psi_{\beta+\gamma}\,x^{\alpha+\gamma}\ , \end{aligned}$$

i.e., using Euler's theorem, we represent the form ψ as a quadratic form in the monomials of degree $k = d/2$, and then we replace the associated matrix by its square. The forms ψ and $Q(\psi)$ are both of the same degree $d = 2k$. The fundamental property of the map $Q : \psi \mapsto Q(\psi)$, which makes it useful for our purposes, is described in the following lemma. The proof can be found in [9].

Lemma 2 *Let ψ be a form of even degree $d = 2k$, and let g be a polynomial of degree $< d$. If the matrix $\Psi_{[k,k]}$ is nonsingular, then $Q(\psi)(x) > 0$ for every nonzero $x \in \mathcal{R}^n$, and the set $\{x : Q(\psi)(x) + g(x) = 0\}$ is either empty or bounded. In particular, for every $\lambda \in \mathcal{R}$, the set $\{x : Q(\psi)(x) + g(x) = \lambda\}$ is either empty or bounded, even for $g \equiv 0$.*

Since $Q(\psi)$ is a form of degree d, we can also write it as in (7)

$$Q(\psi)(x) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!}\,Q_\alpha(\psi)\,x^\alpha\ , \qquad (10)$$

where, $\{Q_\alpha(\psi) : |\alpha| = d\}$ are the coefficients of $Q(\psi)$. Note that for each multiindex α of size d, $Q_\alpha(\psi)$ is a quadratic form in the coefficients of ψ.

5 Two parameterized families of even degree polynomials

As we explained in section 3, forms are related to non-homogeneous polynomials in two different ways. Every polynomial can be written as a sum of forms of different degrees, or, by homogenization, it can be seen as a form in $n+1$ variables. In this section we define two parameterized families of polynomials with bounded level sets based on these two methods.

5.1 A polynomial as a sum of forms

In this case we write a polynomial $f(x)$ of even degree d in n variables as a sum of forms of different degrees, as in equation (8), and parameterize the leading form f_d using the map $Q(\psi)$ defined in the previous section. That is, this family of polynomials is parameterized in the following way

$$f(x) = Q(\psi)(x) + g(x)\ ,$$

where ψ is an arbitrary form of degree d in n variables, and $g(x)$ is a polynomial of degree $< d$ in n variables. The parameterization is quadratic in the coefficients of degree d, and linear in the rest. Finally, Lemma 2 shows that in general, level sets of members of this family are bounded.

5.2 A polynomial as a form

Now we return to non-homogeneous polynomials through the one-to-one correspondence described in equation (9). That is, from now on, multiindices have $n+1$ components $(\alpha_0, \ldots, \alpha_n)$, forms have $n+1$ variables $x_0, \ldots, x_n$, and non-homogeneous polynomials in n variables are obtained from forms through the substitution $x_0 \mapsto 1$.

Since in general the form $Q(\psi)$ has no zeros, we modify the parameterization defined by Q to obtain a parameterized family of polynomials with nonempty, and bounded, zero sets. Since the level sets of $Q(\psi)$ are bounded, we will choose a particular level set, say λ, and the new parameterization will be $Q(\psi) - \lambda$. We choose λ as the minimizer of

$$\frac{\frac{1}{q}\sum_{i=1}^q f(p_i)^2}{\frac{1}{q}\sum_{i=1}^q \|\nabla f(p_i)\|^2}$$

with respect to λ. Since the denominator does not depend on λ, the solution is given by

$$\lambda(\mathcal{D},\psi) = \tfrac{1}{q}\sum_{i=1}^q Q(\psi)(p_i) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!} Q_\alpha(\psi)\,m_\alpha\ ,$$

where m_α is the moment of the data corresponding the the monomial x^α

$$m_\alpha = \tfrac{1}{q}\sum_{i=1}^q (p_i)^\alpha\ ,$$

remembering that $\alpha = (\alpha_0, \ldots, \alpha_n)$ and $x^\alpha = x_0^{\alpha_0}\cdots x_n^{\alpha_n} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ here, because $x_0 = 1$.

In this way we obtain our desired parameterization

$$\phi(\psi, x_1, \ldots, x_n) = \sum_{|\alpha|=d} Q_\alpha(\psi)\,(x^\alpha - m_\alpha)\ . \qquad (11)$$

It parameterizes a family of polynomials which have zero mean value on the data set, and so, non-empty zero sets. The other good property of this parameterization is that it is still quadratic and homogeneous in the coefficients of ψ, and it can be evaluated very inexpensively, as we will see in the next section.

6 Implementation details

Since both parameterized families of polynomials of section 5 are based on the map $Q(\psi)$ of section 4, and our goal is to use them together with the Levenberg-Marquardt algorithm, we need to be able to evaluate as inexpensively as possible the coefficients $\{Q_\alpha(\psi) : |\alpha| = d\}$, and the partial derivatives $\{\partial Q_\alpha(\psi)/\partial\Psi_\beta : |\alpha| = |\beta| = d\}$. Since $Q_\alpha(\psi)$ is a quadratic form in the coefficients $\{\Psi_\beta : \beta\}$ of ψ, we have

$$Q_\alpha(\psi) = \sum_{|\beta|=d}\sum_{|\gamma|=d} Q_{\alpha\beta\gamma}\,\Psi_\beta\Psi_\gamma\ , \qquad (12)$$

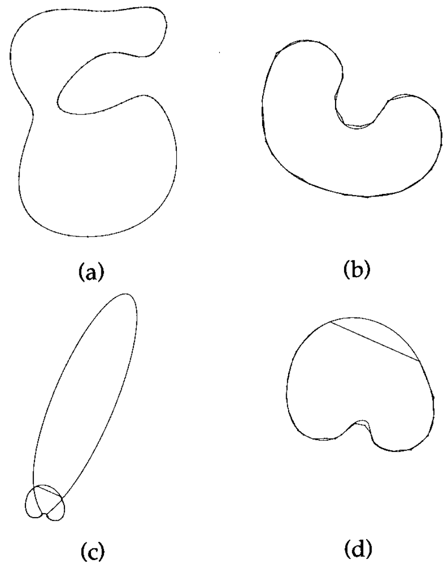

Figure 1: Fitting quartic curves to 2D points (drawn as small circles). See text for details.

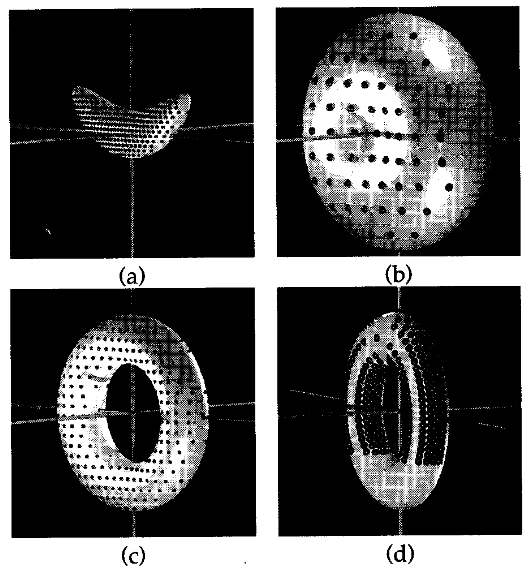

Figure 2: Synthetic surface fitting experiments. (a) A bean. (b) A yoyo. (c) A torus. (d) A torus with data covering only part of the surface.

with $Q_{\alpha\beta\gamma} = Q_{\alpha\gamma\beta}$ for all α, β, γ of size d. The partial derivatives can now be easily derived from this last expression, they are linear functions of the coefficients

$$\frac{\partial Q_\alpha(\psi)}{\partial \Phi_\beta} = 2 \sum_{|\gamma|=d} Q_{\alpha\beta\gamma} \Psi_\gamma \ .$$

In order to evaluate these expressions, we only need to know the numbers $Q_{\alpha\beta\gamma}$. Since these numbers are independent of the form ψ, they can be computed off-line. However, since they are just function of the degree and the space dimension, they can be evaluated at the beginning of the computation, or when the degree changes. Also, since the matrices $\{ [Q_{\alpha\beta\gamma}]_{|\beta|=|\gamma|=d} : |\alpha| = d \}$ are very sparse, for a more efficient evaluation, they should be stored as linked lists. The algorithm for computing the elements of the matrices $[Q_{\alpha\beta\gamma}]$ is based on the defining formulas (10) and (12).

7 Experimental results

We have implemented the method described in section 5.1 and used it to fit quartic curves and surfaces to contour and range data. The Levenberg-Marquardt algorithm requires a set of initial parameters. In our experiments, we have have either set all initial parameters to 1, or used as initial guess the quartic $x^4 + y^4 + z^4 = 1$, suitably translated and scaled to enclose all data points. Both sets of initial parameters have in general led to identical results at convergence.

In the proof of Lemma 2 [9] it is shown that the surface associated with the form ψ is enclosed in a sphere of radius inversely proportional to the minimum eigenvalue of the matrix $\Psi^2_{[2,2]}$. In practice, this means that the eigenvalues of $\Psi^2_{[2,2]}$ give us an indication of the volume enclosed by the surface. To avoid large surface components that may appear where there are gaps in the data, we have chosen in our experiments to divide the error of fit by the trace of $\Psi^2_{[2,2]}$. Since all eigenvalues are positive, this has the effect of maximizing ν, therefore minimizing the enclosed volume.

Figures 1 shows experiments in curve fitting. The data points used in figure 1(a) actually belong to a quartic curve. As shown in the figure, the curve is correctly recovered. Figure 1(b) shows the curve fitted to a polygon entered by hand. Again, a good approximation of the data is obtained. More interestingly, figure 1(c-d) shows another example with a large gap among the data points (once again entered by hand). In figure 1(c), the result of fitting an unconstrained quartic curve is shown: a large component is obtained because of the gap in the data. Figure 1(d) shows the excellent approximation obtained when fitting a bounded, "minimum-area" quartic curve.

Figure 2 shows some experiments in fitting bounded quartic surfaces to synthetic range data. In these experiments, the data points actually belong to a quartic surface: a bean-shaped quartic (figure 2(a)), a yoyo-shaped quartic (figure 2(b)), and a torus (figure 2(c)). The recovered surfaces have been ray traced, and they are displayed along with the coordinate axes and the data points, shown as small spheres. All surfaces have

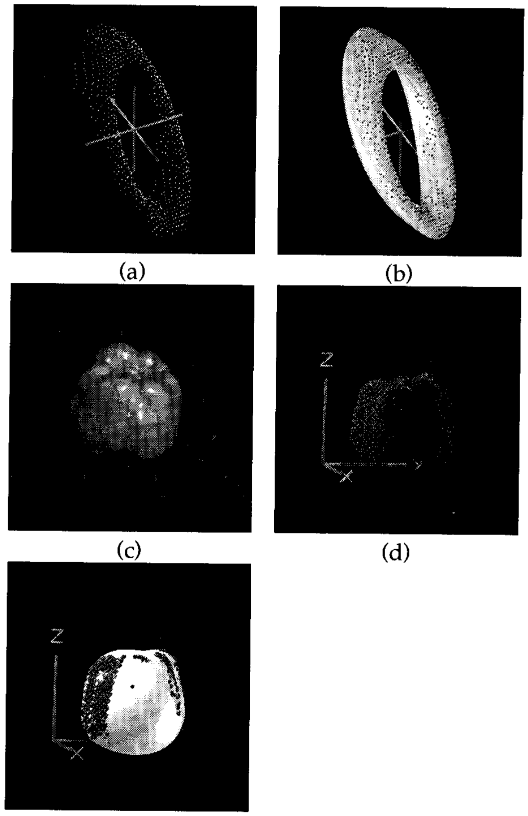

Figure 3: Experiments with real range data. See text.

been correctly recovered, even when the data points only cover about a quarter of the original surface (figure 2(d)).

Figure 3 shows experiments in fitting bounded quartic surfaces to real range data. Figure 3(a) shows the data from a single range image of a torus, and figure 3(b) shows the fitted surface. Less than half of the surface is visible, but a fairly good approximation of the surface is recovered. Figure 3(c-d) shows a pepper and the corresponding range data obtained by registering and merging three range images. Again, despite noise and large gaps in the data, a reasonable surface model is recovered, as demonstrated by figure 3(e).

8 Conclusions

We have described a technique for stabilizing the implicit function fitting process. The key drawback of implicit function fitting methods described in literature thus far has been the unboundedness of the fitted curves and surfaces, or the lack of shape description power. In this paper, trying to solve these two problems, we have introduced two parameterized families of polynomials whose zero sets are always bounded, and with enough flexibility in terms of shape description. Preliminary experimental results with 2D curves and 3D surfaces are encouraging.

In the past, we have used elimination theory to recognize and locate 3D curved objects modelled by parametric algebraic surfaces from monocular image contours [2, 5]. The models were constructed using a CAD system [4]. Here, we have addressed the problem of automatically constructing bounded algebraic surface models by combining several range images. Next, we will attack the problem of automatically constructing bounded algebraic surface models from several video images, using again elimination theory to relate the surface parameters to the observed image contours. We will explore the application of these models to object recognition.

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