Parameterized Families of Polynomials for Bounded Algebraic Curve and Surface Fitting

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***Abstract*— Interest in algebraic curves and surfaces of high degree as geometric models or shape descriptors for different model-based computer vision tasks has increased in recent years, and although their properties make them a natural choice for object recognition and positioning applications, algebraic curve and surface fitting algorithms often suffer from instability problems. One of the main reasons for these problems is that, while the data sets are always bounded, the resulting algebraic curves or surfaces are, in most cases, unbounded. In this paper, we propose to constrain the polynomials to a family with bounded zero sets, and use only members of this family in the fitting process. For every even number d we introduce a new parameterized family of polynomials of degree d whose level sets are always bounded, in particular, its zero sets. This family has the same number of degrees of freedom as a general polynomial of the same degree. Three methods for fitting members of this polynomial family to measured data points are introduced. Experimental results of fitting curves to sets of points in $\mathbb{R}^2$ and surfaces to sets of points in $\mathbb{R}^3$ are presented.**

***Abstract*— Bounded algebraic curves and surfaces, algebraic curve and surface fitting, algebraic invariance.**

I. Introduction

In the past few years several researchers have started using algebraic curves and surfaces of high degree as geometric models or shape descriptors in different model-based computer vision tasks. Typically, the input for these tasks is either an intensity image or dense range data. While early approaches to model-based computer vision focused on polyhedral objects or the blocks world [33], more recent work has considered curved objects that are typically modeled by collections of curved primitives. In nearly all of this work, the primitives have been natural quadrics (spheres, ellipsoids, cylinders, and cones) [8], [12], superquadrics [3], or a well-defined subset of generalized cylinders [5], [11]. Algebraic approaches for recognizing and locating objects represented by implicit (or parametric) polynomial surfaces of degree higher than 2 have only recently been proposed [25], [27], [39], [38].

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One of the fundamental problems in building a recognition and positioning system based on implicit curves and surfaces is how to fit these curves and surfaces to data. This process will be necessary for automatically constructing object models from range or intensity data and for building intermediate representations from observations during recognition. Several methods are available for extracting straight line segments [15], planar patches [17], quadratic arcs [1], [4], [9], [14], [18], [26], [32], and quadric surface patches [7], [8], [12], [17], [21] from 2-D edge maps and 3-D range images. Recently, methods have also been developed for fitting algebraic curve and surface patches of arbitrary degree [13], [25], [28], [27], [37], [39], [38]. This paper primarily addresses the problem of fitting bounded algebraic curves and surfaces to point data. Relatively little work is available on this subject, not even for the particular case of ellipses or ellipsoids. Just recently Keren, Cooper, and Subrahmonia [26] have independently developed similar methods for the limited case of quartic curves and surfaces.

In most work in computer aided design and computer vision, a surface is represented parametrically as a smooth vector function $s : \mathbb{R}^2 \to \mathbb{R}^3$ where each coordinate of s is typically either a polynomial or ratio of polynomials. Examples of parametric surfaces include planes, Bezier patches, nonuniform rational B-splines, and some generalized cylinders. Bicubic patches, which are the most prominent type of surface in computer aided design [16], [36] are given by polynomials with a maximum degree of 3 in each parametric coordinate.

An implicit surface is the set of zeros of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ of three variables:

$$Z(f) = \{(x_1, x_2, x_3)^t : f(x_1, x_2, x_3) = 0\} .$$

Similarly, an implicit 2-D curve is the set $Z(f) = \{(x_1, x_2) : f(x_1, x_2) = 0\}$ of zeros of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ of two variables. The curves or surfaces are *algebraic* if the functions are polynomials. Other common surface representations such as the quadric surfaces (e.g., cones, ellipsoids, hyperboloids, etc.) admit both a parametric and an implicit form. However, there are some algebraic surfaces (third order and higher) that can only be represented implicitly.

Although implicit algebraic curves and surfaces have many good properties that make them the natural choice for object recognition and positioning, parametric curves and surfaces still outperform them in a fundamental area. More stable or robust algorithms are known to approximate sets of measured data points by parametric curves and surfaces than by their implicit counterparts. One of the main problems is that, while the data sets are always bounded, the algebraic curves or surfaces fitted to them are, in most cases, unbounded. Additionally, very small changes in the coefficients of the polynomials often produce very large changes in the global shape of the curve or surface. In general, the notion of locality is partially lost during the fitting process. One possible solution to this problem is to constrain the coefficients of the defining polynomials in such a way that the curves or surfaces defined by these polynomials are always bounded. In general, there are two ways to introduce these constraints. First, constraints on the coefficients can be expressed implicitly by a system of equality and inequality equations. Alternatively, the coefficients of the polynomials can be specified as functions of unbounded parameters; consequently the polynomials are restricted to a family with the desired properties. In this paper, we concentrate on two particular families of parameterized polynomials that leads to certain computational advantages during fitting. A well-known example of a parameterized family of implicit surfaces is the family of superquadrics, which have up to eight shape parameters if bending and tapering are considered (plus six more for pose) [3]. However, eight degrees of freedom leaves little shape control, and something more general is needed.

As an example of one of the presented parameterizations of algebraic curves, consider the case of conics. The only bounded conics are the ellipses. A nonsingular quadratic polynomial in two variables can always be written in matrix form as $f(x_1, x_2) = X^tAX$, where $X = (1, x_1, x_2)^t$ and A is a 3×3 symmetric matrix. Now, every ellipse can be represented as a level set of the polynomial $f(x_1, x_2) = X^tAX$ where A is positive definite. A matrix A is positive definite if and only if $A = BB^t$ for certain nonsingular matrixes B, which can also be taken symmetric. Thus, every ellipse can be represented as a level set of an element of the family of quadratic polynomials $\{f = X^tB^2X : |B| \neq 0\}$. The coefficients of the members of this family are quadratic functions in the elements of the matrix B.

In this paper we generalize the previous construction to higher degree and higher dimensional polynomials. Only polynomials of even degree are considered, because the level sets of polynomials of odd degree always define unbounded curves and surfaces. We show that every polynomial $g(x)$ of even degree $d = 2k$ in n variables $x = (x_1, \cdots, x_n)^t$ can be written in a canonical way as a quadratic form $g = X^tAX$ in the vector $X = (1, x_1, \cdots, x_n^k)^t$ of monomials of degree less than or equal to k, where A is a symmetric matrix and a linear function of the coefficients of g. We then consider the family of polynomials $\{f = X^tB^2X : |B| \neq 0\}$ whose level sets are bounded curves or surfaces. Not every bounded curve or surface of degree d can be represented as a level set of a member of this family, but we show that the family is rich enough in terms of shape description power. In particular, it has as many degrees of freedom as a general polynomial of the same degree.

Fig. 1 shows an example of fitting an unconstrained versus bounded fourth-degree surface to range data. Note that the unbounded surface has a hyperboloid-like shape while the bounded fit leads to a more faithful representation.

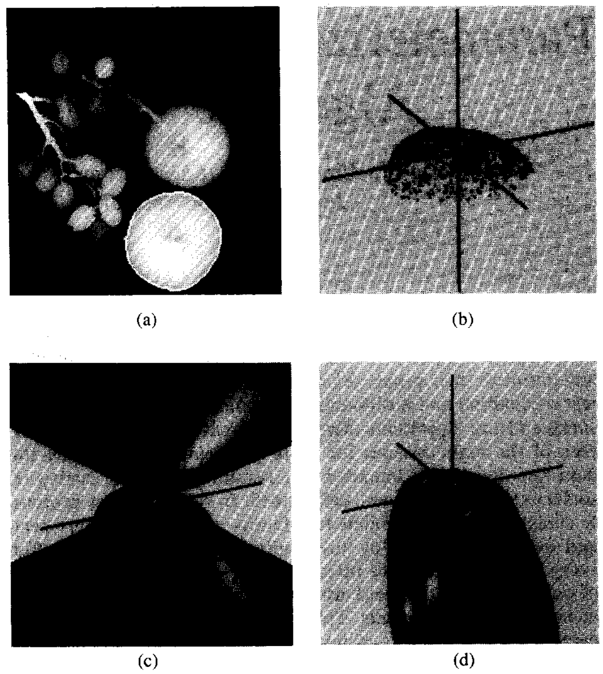


Fig. 1. Unconstrained versus bounded surface fitting to range data. (a) A range image of some fruit from the NRCC range database. (b) Data points represented in 3-D of region FRUI1 in the image. (c) Unconstrained fourth-degree algebraic surface fit. (d) General bounded fourth-degree fit.

The rest of this paper is organized as follows. In Sections II and III we review some previous results, and introduce the notation needed to define the problem that we solve in this paper. A reader familiar with the subject can skip them. In Section IV we establish the connection between positive definite forms, i.e., forms that attain only positive values, and polynomials with bounded level sets. In the same section we show that linear families of polynomials with bounded level sets are quite limited, that at least a quadratic parameterization is needed, and that a quadratic parameterization is rich enough. In Section V we introduce Euler's theorem. Euler's theorem lets us write, in a canonical way, a form of even degree d as a quadratic form in the monomials of degree $d/2$. In Section VI we use Euler's theorem to construct a canonical quadratic parameterization of positive definite forms and prove several important properties of the parameterization. In Section VII we use the previous parameterization of positive definite forms—and the two different representations of polynomials discussed above—to construct two parameterized families of polynomials with bounded sets of zeros. Experimental results of fitting curves and surfaces to 2-D and 3-D point data, respectively, using one of the parameterizations are presented in Section VIII. We conclude with a brief discussion of some future research directions in Section IX. Finally, to improve the readability of the paper we have moved most of the proofs to the Appendix.

II. Algebraic Curve and Surface Fitting

In this section we define what we mean by a parameterized family of polynomials. Such a family in turn defines a parameterized family of algebraic curves or surfaces. We then review some previous results on fitting algebraic curves and surfaces to measured data points (i.e., < selecting the member of the family that is nearest to the data set), and finally discuss a necessary property (covariance) of these families.

A. Parameterized Families of Polynomials

The first step of the fitting process is to restrict the defining polynomials to a family parameterized by a finite number of parameters. Let $\phi : \mathbb{R}^{r+n} \to \mathbb{R}$ be a smooth function, and let us consider the maps $f : \mathbb{R}^n \to \mathbb{R}$, which can be written as

$$f(x) \;=\; \phi_u(x) \;\equiv\; \phi(u,x)$$

for certain $u = (u_1, \cdots, u_r)^t \in \mathbb{R}^r$. We will refer to $u_1, \cdots, u_r$ as the *parameters* and to $x_1, \cdots, x_n$ as the *variables*. The family of all such maps will be denoted

$$\mathcal{F}_\phi \;=\; \{f : \exists u \; f = \phi_u\} \;,$$

and we will say that ϕ is the *parameterization* of the family $\mathcal{F}_\phi$. The set of zeros $Z(f)$ of a member f of $\mathcal{F}_\phi$ is a 2-D curve when $n = 2$ and a surface when $n = 3$. For the rest of the paper we will impose a further restriction on the parameterization ϕ; we will request that members of $\mathcal{F}_\phi$ be polynomials of degree $\leq d$, for certain positive integers d.

As an example, consider the family of polynomials given by

$$\phi(u,x) \;=\; x_1^2 + x_2^2 - u^2.$$

In this case, there is one parameter u and two variables x_1, x_2; clearly, the zero set of a member of this family is simply a circle centered at the origin, and the one shape parameter determines its radius.

An important class of parameterizations, which corresponds to several cases of algebraic curves or surfaces, is the linear model. In the linear model the maps can be written as follows:

$$\phi(u,x) = u_1X_1(x) + \cdots + u_rX_r(x) = FX(x), \quad (1)$$

where $F = (u_1, \cdots, u_h)$ is a row vector of coefficients, the transpose of the parameter vector, and $X = (X_1, \cdots, X_r)^t : \mathbb{R}^n \to \mathbb{R}^r$ is a fixed map. We will say that a parameterization of this kind is a *linear parameterization*, and the corresponding family a *linear family*. For example, the family of all polynomials of degree $\leq d$ in n variables can be written as a linear family. Just take X as the vector of monomials of degree $\leq d$. For example, for $n = 2$ and $d = 2$ we can take

$$X(x_1,x_2) = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} .$$

As will be seen in subsequent sections, the linear model is important because fitting is quite simple. Unfortunately, linear families of polynomials with bounded zero sets are quite limited, and at least a quadratic parameterization will be needed.

B. Fitting

Given a finite set of n-dimensional ($n = 2$ or $n = 3$) data points $\mathcal{D} = \{p_1, \cdots, p_q\}$, the problem of fitting an implicit curve or surface $Z(f)$ to the data set $\mathcal{D}$ corresponds to determining the $\hat{f} \in \mathcal{F}_\phi$ that minimizes the mean square distance

$$\frac{1}{q}\sum_{i=1}^{q} \operatorname{dist}^2(p_i, Z(f)) \quad (2)$$

from the data points to the curve or surface $Z(f)$.

Unfortunately, there is no closed form expression for the distance from a point to a generic implicit curve or surface, not even for algebraic curves or surfaces. This has motivated a number of approximations to the exact distance.

The simplest approach to fitting is based on the *algebraic* distance

$$\operatorname{dist}^2(x, z(f)) \approx f(x)^2,$$

because without noise $f(x)$ vanishes for all x on $Z(f)$. For linear families of polynomials, computing the surface coefficients is thus reduced to an eigenvalue problem. Although computationally attractive, fitting based on this distance may be biased and is not covariant, as discussed in Section III–C.

Taubin based his algebraic curve and surface fitting algorithms [37], [39] on the first order approximation

$$\operatorname{dist}^2(x, Z(f)) \;\approx\; \frac{f^2(x)}{\|\nabla f(x)\|^2} \; . \quad (3)$$

The mean value of the function (3) on a fixed set of data points is a smooth nonlinear function of the parameters, and can be *locally* minimized using well-established nonlinear least squares techniques. However, since one is interested in the *global minimum* and wants to avoid a global search, a second approximation

$$\frac{\frac{1}{q}\sum_{i=1}^{q} f^2(p_i)}{\frac{1}{q}\sum_{i=1}^{q} \|\nabla f(p_i)\|^2} \;=\; \frac{FMF^t}{FNF^t} \;, \quad (4)$$

is used to turn the choice of initial parameters for linear families into a generalized eigenvalue problem. More recently, Taubin developed a more accurate approximation for graphics applications [40], [42], [43], which he also used for algebraic curve and surface fitting [41], yielding better results than with (3). This method directly applies to fitting unconstrained algebraic curves and surfaces, because the family of polynomials of degree $\leq d$ in n variables is a linear family. Later on we will show the generalized eigenvalue fit method is related to bounded surfaces as well.

Finally, Kriegman and Ponce [27], [30] have shown that elimination theory can be used to construct a closed-form expression for the *exact distance* from a point $x \in \mathbb{R}^n$ to the zero set $Z(f)$ of a polynomial.

For some applications, the elimination of n variables followed—at every step of the minimization and for each data point—by the computation the roots of D may be computationally impractical (see [39] for an alternative approach based on constrained optimization). In the experiments described in

Section X, we have either minimized the algebraic distance or used Taubin's method to fit bounded algebraic curves and surfaces to 2-D and 3-D data.

C. Covariance Properties of Parameterized Families

Any one of the previous fitting algorithms can be seen as an operator Λ, which assigns a set of data points $\mathcal{D} = \{p_1, \cdots, p_q\}$ to a member f of the parameterized family $\mathcal{F}_\phi$

$$\mathcal{D} \mapsto f = \Lambda(\mathcal{D}) \in \mathcal{F}_\phi .$$

Since we want to use the results of our fitting algorithms for object recognition, our main concern is viewpoint independence, or, to be more precise, *covariance*. That is, if $x' = Rx + t$ is a rigid body transformation (i.e., $R \in SO(n)$ is a rotation in $\mathbb{R}^n$, and $t \in \mathbb{R}^n$ is a translation) which represents a change of coordinate systems, and we denote $\mathcal{D}' = \{p'_1, \cdots, p'_q\}$, where $p'_i = Rp_i + t$ are the coordinates of the points in the new coordinate system, we expect $f' = \Lambda(\mathcal{D}')$ and f to be related as follows: $f'(x') = f(x)$. Thus, if two range images of a set of points were acquired from two locations, the two resulting surfaces should only differ by a rotation and translation. This requirement imposes a restriction on the parameterized family. If f is a member of $\mathcal{F}_\phi$, for every rigid body transformation $x' = Rx + t$ the function $f(R^t(x - t))$ should also be a member of $\mathcal{F}_\phi$. That is, if $f(x) = \phi(u, x)$ for the parameter vector u, there must exist another parameter vector u' such that $f(R^t(x-t)) = \phi(u', x)$. In other words, the family $\mathcal{F}_\phi$ has to be closed under the action of the motion group.

When the family is not closed under the action of the motion group, one can always enlarge the family by introducing new explicit transformation parameters. If $\phi(u, x)$ is the parameterization of the family that is not closed under the action of the Euclidean group, we can look at the expression $\phi(u, Rx+t)$ as a function of the parameters (u, R, t) and the space variable x, where the rotation matrix R can also be parameterized using, for example, the three Euler angles. A typical example of this procedure is the case of superquadrics [3], [22], [37], [38]. But the fitting algorithms become more complex. It is better to use families that are already closed under rotation and translation. If an object-centered coordinate system is required, in the case of polynomials it can be computed from the coefficients after fitting [44].

Of course, the closure of the family under rotation and translation is not sufficient to guarantee the covariance of the fitting algorithms, but it is clearly necessary. For example, the family of unconstrained polynomials of a given degree is closed under even more general transformations—projective, to be precise—and the fitting algorithms described above are covariant under similarity transformations (rotation, translation, and scale) [39], [38].

We will study the closure properties of families of polynomials with bounded zero sets at the end of Section IV.

III. Polynomials and Forms

In this section, a few different notations will be introduced to represent multivariate polynomials of degree d in n variables. From now on, polynomials will be written expanded in Taylor series at the origin

$$f(x) = f(x_1, \cdots, x_n) = \sum_\alpha \tfrac{1}{\alpha!} F_\alpha x^\alpha , \qquad (5)$$

where the vector of nonnegative integers $\alpha = (\alpha_1, \cdots, \alpha_n)^t$ is a *multiindex* of *size* $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ is a multiindex factorial, $F_\alpha \in \mathbb{R}$ is a coefficient of degree $|\alpha|$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a *monomial* of degree $|\alpha|$. There are exactly $h_d = \binom{n+d-1}{n-1} = \binom{n+d-1}{d}$ different multiindexes of size d, and therefore the same number of monomials of degree d. A polynomial of degree d in n variables has $h_d + h_{d-1} + \cdots + h_0 = \binom{n+d}{n}$ coefficients, as many as a homogeneous polynomial of the same degree but in $n+1$ variables. For example, a polynomial defining a quartic surface ($n = 3, d = 4$) has 35 coefficients, while a sextic surface ($n = 3, d = 6$) is defined by 84 coefficients. Since a polynomial is homogeneous in its coefficients and six parameters determine the position and orientation of a surface, quartic and sextic surfaces are defined by 28 and 77 shape parameters, respectively.

If all the elements of a parameterized family $\mathcal{F}_\phi$ are polynomials of degree $\leq d$, we will also write

$$\phi(u, x) = \sum_{0 \leq |\alpha| \leq d} \tfrac{1}{\alpha!} F_\alpha(u) x^\alpha .$$

where $F_\alpha(u)$ is a scalar valued function in the parameters u. A polynomial $\psi(x)$ is homogeneous (called a *form*) if all its terms are of the same degree

$$\psi(x) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!} \Psi_\alpha x^\alpha . \qquad (6)$$

where $\Psi_\alpha \in \mathbb{R}$. Equivalently, a polynomial $\psi(x)$ is a form of degree d if and only if $\psi(\theta x) = \theta^d \psi(x)$. This will prove useful in the proofs of some of the forthcoming lemmas.

Forms are related to nonhomogeneous polynomials in two ways. Both methods will be used to define different parameterizations of families of polynomials with bounded level sets in section VII.

A. A Polynomial as a Sum of Forms

In the first place, every polynomial $f(x)$ of degree d can be written in a unique way as a sum of forms

$$f(x) = \sum_{k=0}^{d} \psi_k(x) , \qquad (7)$$

where $\psi_k(x)$ is a form of degree k and $\psi_d(x) \not\equiv 0$. The form ψ_d is called the *leading form* of the polynomial f.

B. A Polynomial as a Form

In the second place, by introducing homogeneous coordinates, every curve or surface described in Euclidean space by a polynomial in n variables can be described in projective space by an associated form in $n+1$ variables [49]. If $\psi(x_0, \cdots, x_n)$ is a form of degree d in $n+1$ variables, and $f(v_1, \cdots, v_n)$ is a (nonhomogeneous) polynomial of degree $\leq d$ in n variables, the one-to-one correspondence is given by

$$\begin{array}{rcl} \psi(x_0, x_1, \cdots, x_n) & = & f(x_1/x_0, \cdots, x_n/x_0)x_0^d \\ f(v_1, \cdots, v_n) & = & \psi(1, v_1, \cdots, v_n) \, . \end{array} \quad (8)$$

In other words, every polynomial in n variables is the restriction of a form in $n+1$ variables to the hyperplane $\{x : x_0 = 1\}$, and every form in $n+1$ variables is totally determined by its restriction to this hyperplane.

C. Monomials and Vector Notation

We now introduce a more compact vector notation for representing polynomials by introducing a linear ordering of multiindexes; this will allow us to represent a form as the product of a vector of coefficients $\Psi_{[d]}$ and a vector of monomials $X_{[d]}(x)$. Although there are many possible orderings, we will use only the *lexicographical order* (defined in the Appendix); the same results can be obtained using other orders. The set of monomials $\{x^\alpha/\sqrt{\alpha!} : |\alpha| = d\}$ of degree d, lexicographically ordered, define a vector of dimension h_d, which we will denote as $X_{[d]}(x)$. For example,

$$X_{[3]}(x_1, x_2) = \begin{pmatrix} \frac{1}{\sqrt{6}} x_1^3 \\ \frac{1}{\sqrt{2}} x_1^2 x_2 \\ \frac{1}{\sqrt{2}} x_1 x_2^2 \\ \frac{1}{\sqrt{6}} x_2^3 \end{pmatrix}^t .$$

The reason for the introduction of the square roots will be evident later on, but with them there is a nice relation between the Euclidean norm of $X_{[d]}$ and the Euclidean norm of x.

Lemma 1: If x and y are two vectors of independent variables, then $X_{[d]}(x)^t X_{[d]}(y) = \frac{1}{d!}(x^t y)^d$. In particular, $\|X_{[d]}(x)\|^2 = \frac{1}{d!}\|x\|^{2-D}$.

Note that most of the proofs appear in the Appendix. Consistent with this notation, the vector $\{\Psi_\alpha/\sqrt{\alpha!} : |\alpha| = d\}$ of coefficients of ψ will be denoted $\Psi_{[d]}$. In this way, a form ψ of degree d can be written in vector notation as

$$\psi(x) = \Psi_{[d]}^t X_{[d]}(x) \, . \quad (9)$$

Finally, for every pair of indexes j, k, the rank one matrix of monomials $X_{[j]}(x) X_{[k]}(x)^t$ will also be denoted $X_{[j,k]}(x)$.

IV. Positive Definite Forms

There is an intimate relation between polynomials with bounded zero sets and algebraic inequalities. In this chapter we study the relation between positive definiteness of the defining polynomial and boundedness of the corresponding set of zeros. We start with forms.

A form ψ is *positive definite* if $\psi(x) > 0$ for all $x \neq 0$ (by homogeneity $\psi(0) = 0$), and *nonnegative definite* (sometimes called positive semidefinite) if $\psi(x) \geq 0$ for all $x \in \mathbb{R}^n$. *Negative definite* and *nonpositive definite* forms can be defined in a similar way. A form will be called *definite* if it belongs to one of the previous four categories, and *indefinite* otherwise. Since the map $\psi(x) \mapsto -\psi(x)$ establishes a one-to-one correspondence between nonnegative and nonpositive definite forms, it is sufficient to study the first two classes. The first important result is that only nonzero forms of even degree can be definite.

Lemma 2: There exist no nonzero definite forms of odd degree.

For every form ψ of degree d, the two constants

$$\begin{array}{rcccl} \psi_{\text{MIN}} & = & \min\limits_{\|x\|=1} \psi(x) & = & \min\limits_{x \neq 0} \dfrac{\psi(x)}{\|x\|^d} \\ \psi_{\text{MAX}} & = & \max\limits_{\|x\|=1} \psi(x) & = & \max\limits_{x \neq 0} \dfrac{\psi(x)}{\|x\|^d} \end{array}$$

are finite, because the unit ball $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is compact and ψ is continuous, and by definition they satisfy the following inequality:

$$\psi_{\text{MIN}} \|x\|^d \leq \psi(x) \leq \psi_{\text{MAX}} \|x\|^d \quad (10)$$

for every $x \neq 0$. Clearly, ψ is positive definite if and only if $\psi_{\text{MIN}} > 0$, and if ψ_1 and ψ_2 are forms of the same degree, then $(\psi_1 + \psi_2)_{\text{MIN}} \geq (\psi_1)_{\text{MIN}} + (\psi_2)_{\text{MIN}}$. Finally, it is not difficult to show that every form of even degree can be written as the difference of two positive definite forms of the same degree, but we will omit the proof.

A. Level Sets of Positive Definite Forms are Bounded

For every function f the set $Z_\lambda(f) = \{x \in \mathbb{R}^n : f(x) = \lambda\}$ will be called *the set of level* λ of f, and the elements of $Z_\lambda(f)$ *points of level* λ of f. Our interest in positive definite forms is partially due to the following result.

Lemma 3: The level sets of a positive definite form are either bounded or empty.

Proof: Let ψ be a positive definite form of degree d, and let $\lambda \in \mathbb{R}$. If $\lambda < 0$, clearly $Z_\lambda(\psi) = \emptyset$. If $\lambda = 0$ we have $Z_0(\psi) = \{0\}$, because $\psi(0) = 0$ by homogeneity, and $\psi(x) > 0$ for $x \neq 0$ by hypothesis. If $\lambda > 0$, we have

$$\psi_{\text{MAX}} \|x\|^d \geq \lambda = \psi(x) \geq \psi_{\text{MIN}} \|x\|^d > 0$$

and so

$$0 < \left(\frac{\lambda}{\psi_{\text{MAX}}}\right)^{1/d} \leq \|x\| \leq \left(\frac{\lambda}{\psi_{\text{MIN}}}\right)^{1/d} .$$

Note that all the sets of level $\lambda > 0$ of a positive definite form are in one-to-one correspondence with each other. In fact, for each $\lambda > 0$, the map $S^{n-1} \to Z_\lambda(\psi)$ given by $x \mapsto (\lambda/\psi(x)^{1/d})x$, defines a diffeomorphism between the sphere and the level set $Z_\lambda(\psi)$. For example, Fig. 2 shows the sets of level $1/8$, $1/2$, and 2 of the form $\psi(x_1, x_2) = x_1^4 - 3x_1^2 x_2^2 + 3x_3^4$.

The converse of Lemma 3 is also true.

Lemma 4: The level sets of a nonpositive definite form are either empty or unbounded.

If we now look at a polynomial f in n variables as the restriction of a positive definite form ψ in $n+1$ variables to the hyperplane $\{x \in \mathbb{R}^{n+1} : x_0 = 1\}$, as in (12), we conclude that the level sets of the polynomial f, being the intersection of a hyperplane with a bounded set, are also either bounded or empty. For example, Fig. 3 shows the set of level $3/2$ of the form $\psi(x_0, x_1, x_2) = 2x_0^4 - 3x_0^2 x_1^2 + 3x_1^4 - 3x_1^2 x_2^2 + 6x_2^4 - 2x_0^3 x_2$.

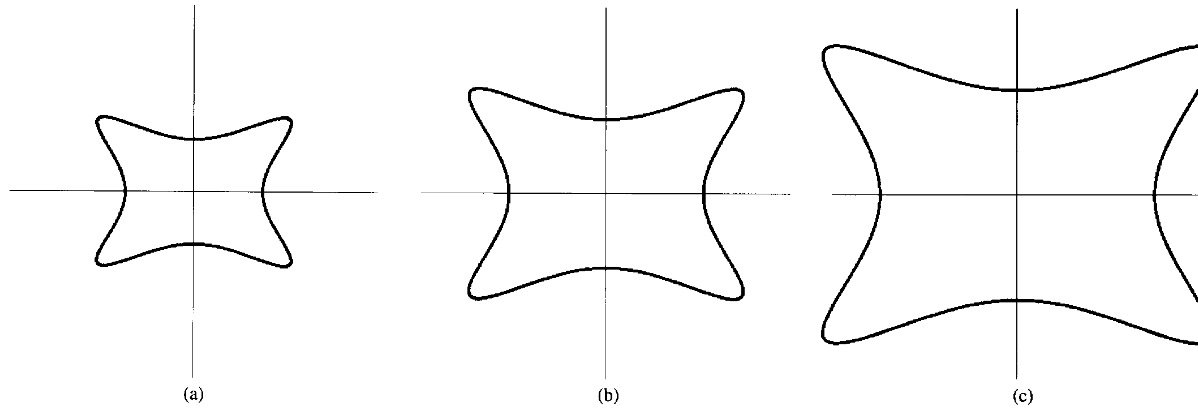


Fig. 2. Level sets of the positive definite form $\psi(x_1, x_2) = x_1^4 - 3x_1^2x_2^2 + 3x_2^4$. (a) Level 1/8. (b) Level 1/2. (c) Level 2.

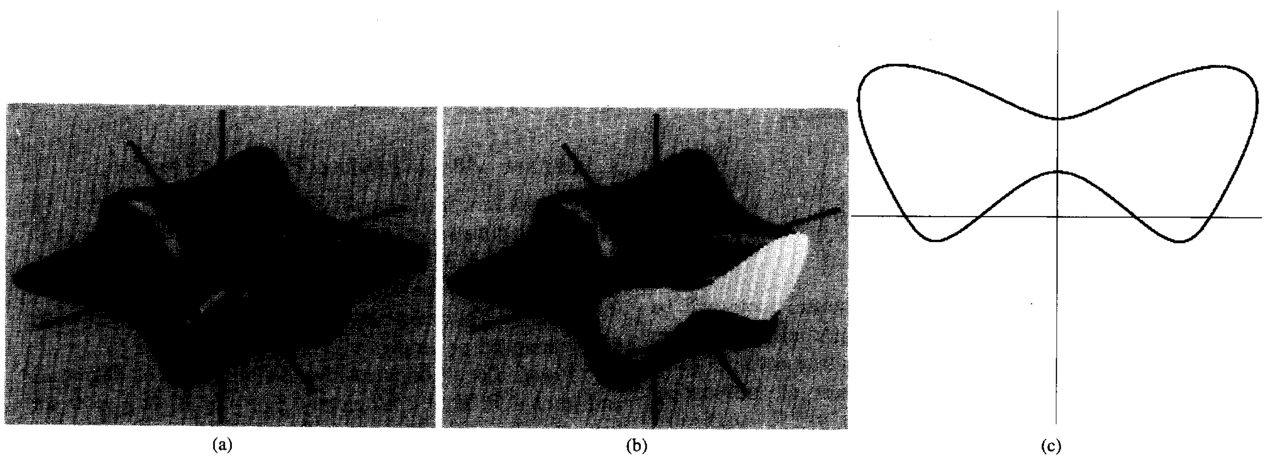


Fig. 3. (a) Set of level 3/2 of the positive definite form $\psi(x_0, x_1, x_2) = 2x_0^4 - 3x_0^2x_1^2 + 3x_1^4 - 3x_1^2x_2^2 + 6x_2^4 - 2x_0^3x_2$. (b) Intersection of the halfspace $\{x : x_0 \leq 1\}$ with the level set of (a). (c) Bounded intersection curve.

Note that although positive definiteness is a sufficient condition for boundedness here, it is clearly not necessary. For example, the form $\psi(x_0, x_1, x_2) = x_1^2 + x_2^2 - x_0^2$ is not positive definite, but $f(x_1, x_2) = \psi(1, x_1, x_2) = x_1^2 + x_2^2 - 1$ is a polynomial with bounded level sets. That is, the restriction of a nonpositive definite form to the hyperplane $x_0 = 1$ can be bounded.

B. Polynomials with a Positive Definite Leading Form

If we now look at a polynomial f in n variables as a sum of forms of different degrees, as in Section IV–A, the positive definiteness of the leading form implies boundedness of the level sets, and indefiniteness implies unboundedness.

Lemma 5: The level sets of a polynomial with a positive definite leading form are either bounded or empty.

While a formal proof is given in the Appendix, using projective geometry it can be seen that for the zero set of a polynomial to be unbounded it must have a point at infinity. Since points at infinity correspond to the leading form vanishing for some nonzero point, a positive definite leading form implies that there are no points at infinity; consequently the level sets are necessarily bounded.

Lemma 6: The level sets of a polynomial with an indefinite leading form are unbounded and never empty. In particular, level sets of odd-degree polynomials are unbounded and never empty.

Note that if the leading form is nonnegative definite but not positive definite, the boundedness depends on the lower degree forms. For example, the level sets of $f(x_1, x_2) = x_1^4 + x_2^3$ are unbounded, but the level sets of $f(x_1, x_2) = x_1^4 + x_2^2$ are bounded.

C. Families of Polynomials and Positive Definiteness

The next issue to be considered is how to parameterize the family of positive definite forms of a given degree. Ideally, we would like to find a linear parameterization, but unfortunately the linear families of definite forms are quite limited, as the following lemma shows.

Lemma 7: If ψ_1 and ψ_2 are two linearly independent forms of the same degree, there exist constants μ_1 and μ_2 such that the form $\mu_1\psi_1 + \mu_2\psi_2$ is indefinite.

It follows that a linear family of definite forms has dimension at most 1. And if $\mathcal{F}_\phi$ is a linear family of polynomials of degree $\leq d$, all of them with definite form of degree d, the subspace of forms of degree d of members of $\mathcal{F}_\phi$, being itself a linear family of definite forms, must be one-dimensional. In other words, in the linear model (see (1)), the *only* way to obtain polynomials with definite leading form is to choose X_1 as a polynomial of degree d with positive definite leading form, and $X_2, \cdots, X_r$ as polynomials of degree $< d$. For example, we can take X_1 as a positive definite form, and $X_2, \cdots, X_r$ as all the monomials of degree $< d$. In this way the resulting linear family turns out to be closed under translation. If we also want this family to be closed under rotations as well, there is basically only one choice, i.e., $X_1(x) = \|x\|^d$, because this is the only form invariant under rotation.

Lemma 8: Let $\psi(x)$ be a form of degree d invariant under orthogonal coordinate transformations. If d is odd, then ψ is identically zero. If d is even, then $\psi(x) = \kappa\|x\|^d$ for certain constant κ.

We can apply the generalized eigenvalue fit method to these linear families, and since in general the coefficient of X_1 will be nonzero, the fitted curve or surface will be bounded. We can also impose as a constraint that the coefficient of X_1 be equal to 1. In this case the fitting can be reduced to solving a linear regression problem [13], [31].

D. Forms that are Sums of Squares

Since linear parameterizations of definite forms are not very powerful, we now look at quadratic parameterizations, i.e., where the coefficients of the forms are quadratic functions of the parameters. The square of an arbitrary form of degree k is a nonnegative form of degree $d = 2k$. A sum of several squares of forms of degree k is also a nonnegative form of degree $d = 2k$

$$\psi = \xi_1^2 + \cdots + \xi_h^2 \,. \tag{11}$$

The coefficients of ψ are quadratic functions of the coefficients of $\xi_1, \cdots, \xi_h$, which can be seen as the parameters. One naturally wonders whether the family of all positive definite forms can be parameterized in this form or not. Unfortunately, not every nonnegative form with real coefficients can be represented as a sum of squares of forms. Hilbert's 17th problem, solved by Artin [2], establishes that every such form can be represented as a sum of squares of rational functions. A theorem by Hilbert [23] gives necessary and sufficient conditions on the degree of the forms, d, and the dimension of the space, n, for every nonnegative form with real coefficients to be representable as a sum of squares of forms. Explicitly, the conditions are $n \leq 2$, or $d = 2$, or $(n, d) = (3, 4)$. For other values of (n, d) counterexamples can be constructed [6, chapter 6], [31]. Although in general not every nonnegative form with real coefficients can be represented as a sum of squares of forms, the family of forms that can be represented as sums of squares is very rich indeed, as we will see in subsequent sections.

To define a parameterization for the family of forms that can be represented as sums of squares, we need to know how many terms are sufficient in the sum. For that we have the following result.

Lemma 9: Every form of degree $d = 2k$ that can be represented as a sum of an arbitrary number of squares of forms of degree k can also be represented as a sum of at most h_k squares of forms of degree k.

The expression $\psi(x) = X_{[k]}^t M X_{[k]}$, with M symmetric and positive definite, defines a parameterization of the family of forms of degree $2k$, which can be written as sums of squares. The main problem with this parameterization is that there are many more degrees of freedom in the matrix M than coefficients of the form ψ, and so every form has an infinite number of representations of this kind. This overparameterization can be a problem for many numerical minimization algorithms. The solution is to restrict the matrix M to have a particular structure with fewer degrees of freedom, i.e., where the matrix M itself is parameterized with fewer parameters. In the next section we show a canonical way to do so.

V. Euler's Theorem

In this section we show how to write a polynomial of even degree d as a quadratic form in the monomials of degree $d/2$ in a canonical way, as described in the previous section, with the matrix M being a linear function of the coefficients of the form. This representation will lead us to a very attractive parameterized family of positive definite forms.

Let ψ be a form of degree d. For every pair of integers, (j, k) such that $j + k = d$, the set of coefficients

$$\left\{ \sqrt{\tfrac{1}{\alpha!\,\beta!}}\ \Psi_{\alpha+\beta} : |\alpha| = j, |\beta| = k \right\}$$

lexicographically ordered in both indexes, defines an $h_j \times h_k$ matrix, which we will denote $\Psi_{[j,k]}(x)$. Now we have all the necessary elements to state Euler's theorem in the form we need. The classical proof for $j = 1$ can be found in [49]. For completeness, the proof of the general case is included in the Appendix.

Lemma 10 (Euler's theorem): For every form ψ of degree $d = j + k$, we have

$$\begin{aligned}\tbinom{d}{j}\psi(x) &= X_{[j]}^t(x)\Psi_{[j,k]}X_{[k]}(x)\\ &= \sum_{|\alpha|=j}\sum_{|\beta|=k} \tfrac{1}{\alpha!\beta!}\Psi_{\alpha+\beta}x^{\alpha+\beta}\,.\end{aligned}$$

In particular, every form ψ of even degree $d = 2k$ can be written as a quadratic form in the monomials of degree $k = d/2$:

$$\begin{aligned}\tbinom{d}{k}\psi(x) &= X_{[k]}^t(x)\Psi_{[k,k]}X_{[k]}(x)\\ &= \sum\sum_{|\alpha|=|\beta|=k} \tfrac{1}{\alpha!\beta!}\Psi_{\alpha+\beta}x^{\alpha+\beta}\,.\end{aligned} \tag{12}$$

VI. A Parameterized Family of Positive Definite Forms

For every form ψ of even degree $d = 2k$, let

$$\begin{aligned}Q(\psi)(x) &= X_{[k]}^t(x)\Psi_{[k,k]}^2X_{[k]}(x)\\ &= \sum\sum\sum_{|\alpha|=|\beta|=|\gamma|=k} \tfrac{1}{\alpha!\beta!\gamma!}\Psi_{\alpha+\beta}\Psi_{\beta+\gamma}x^{\alpha+\gamma}\,,\end{aligned}$$

i.e., using Euler's theorem, we represent the form ψ as a quadratic form in the monomials of degree $k = d/2$, and then we replace the associated matrix by its square. The forms ψ and $Q(\psi)$ are both of the same degree $d = 2k$. The fundamental property of the map $Q : \psi \mapsto Q(\psi)$, which makes it useful for our purposes, is described in the corollary to the following lemma.

Lemma 11: Let ψ be a form of even degree $d = 2k$. If the matrix $\Psi_{[k,k]}$ is nonsingular, then $Q(\psi)$ is positive definite.

Corollary: If ψ is a form of even degree $d = 2k$, g is a polynomial of degree $< d$, and the matrix $\Psi_{[k,k]}$ is nonsingular, then all the level sets of the polynomial $f = Q(\psi) + g$ are either empty or bounded.

Since $Q(\psi)$ is a form of degree d, we can also write it as in (10):

$$Q(\psi)(x) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!} Q_\alpha(\psi) x^\alpha , \tag{13}$$

where $\{Q_\alpha(\psi) : |\alpha| = d\}$ are the coefficients of $Q(\psi)$. Note that for each multiindex α of size d, $Q_\alpha(\psi)$ is a quadratic form in the coefficients of ψ.

The quadratic map $Q : \psi \mapsto Q(\psi)$ has two other properties that make it even more useful for our purposes. We will describe them here, but since the proofs are long and not necessary for understanding the rest of the paper, we have placed them in the Appendix. The first of these properties is the covariance under rotations.

Lemma 12: If $x' = Rx$ is an orthogonal coordinate transformation, and $\psi'(x') = \psi(R^t x')$, then $Q(\psi)' = Q(\psi')$

That is, it is exactly the same to change the coordinate system first and then apply Q as it is to first apply Q and then change the coordinate system. The second property is really necessary for the good behavior of the Levenberg–Marquardt algorithm.

Lemma 13: Seen as a map $\mathbb{R}^{h_d} \to \mathbb{R}^{h_d}$, i.e., $\{\Psi_\alpha : |\alpha| = d\} \mapsto \{Q_\alpha(\psi) : |\alpha| = d\}$, the map Q is locally one to one, except for a subset of forms with coefficients satisfying a certain algebraic equation.

VII. Two Parameterized Families of Even-Degree Polynomials

As we explained above, forms are related to nonhomogeneous polynomials in two different ways. Every polynomial can be written as a sum of forms of different degrees, or as the restriction of a form in $n + 1$ variables to the hyperplane $x_0 = 1$. In this section we define two parameterized families of polynomials with bounded level sets based on these two representations.

A. A Polynomial as a Sum of Forms

In this case we write a polynomial f of even degree d in n variables as a sum of forms of different degrees (11), and parameterize the leading form f_d using the map $Q(\psi)$ defined in the previous section and the other forms linearly. That is, this family of polynomials is parameterized in the following way:

$$f(x) = Q(\psi)(x) + g(x) ,$$

where ψ is an arbitrary form of degree d in n variables and $g(x)$ is a polynomial of degree $< d$ in n variables. The parameterization is quadratic in the coefficients of degree d and linear in the rest.

The generalized eigenvector fit method can also be used in this case to initialize the fitting algorithm as in the unconstrained case. Here we use the linear family parameterized as follows:

$$f(x) = u_1 \|x\|^d + g(x) ,$$

where u_1 is the first parameter and $g(x)$ is a polynomial of degree $< d$ in n variables. After this, and in order to initialize the nonlinear least squares minimization, we still need to compute the coefficients of the form ψ such that $Q(\psi) = \|x\|^d$; fortunately, $\psi(x) = \|x\|^d$ is an eigenfunction of the operator Q.

B. A Polynomial as a Form

Now we return to nonhomogeneous polynomials through the one-to-one correspondence described in (12). That is, in this section multiindexes have $n{+}1$ components $(\alpha_0, \cdots, \alpha_n)$, forms have $n{+}1$ variables $x_0, \cdots, x_n$, and nonhomogeneous polynomials in n variables are obtained from forms through the substitution $x_0 \mapsto 1$.

Since in general the form $Q(\psi)$ has no zeros other than $x = 0$, we modify the parameterization defined by Q to obtain a parameterized family of polynomials with nonempty, and bounded, zero sets. Since the level sets of $Q(\psi)$ are bounded, we can choose a particular level set, say λ, and the new parameterization will be $Q(\psi) - \lambda$. The problem is that we cannot take a fixed value of λ, because for many forms, $Q(\psi) - \lambda$ will still have empty zero set. Since we ultimately want to use this parameterization to estimate an approximation of a data set $\mathcal{D} = \{p_1, \cdots, p_q\}$ by nonlinear least squares methods, we can obviously include λ as a new free parameter in the parameterization of the family of polynomials to be used; however, a better solution is to make λ a function of the data set and the coefficients of ψ, i.e., the parameters. That is, we are looking for a function $\lambda(\mathcal{D}, \psi)$ such that the level set $\{(x_1, \cdots, x_n)^t : Q(\psi)(1, x_1, \cdots, x_n) = \lambda(\mathcal{D}, \psi)\}$ best approximates the set $\mathcal{D}$ in certain sense. We can choose it as the minimizer of the approximate square distance with respect to λ, where $f(x_1, \cdots, x_n) = Q(\psi)(1, x_1, \cdots, x_n) - \lambda$, but this gives an expression that is expensive to evaluate. Instead, we choose λ as the minimizer of (4) with respect to λ. Since the denominator does not depend on λ, the solution is given by

$$\lambda(\mathcal{D}, \psi) = \tfrac{1}{q} \sum_{i=1}^{q} Q(\psi)(p_i) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!} Q_\alpha(\psi) m_\alpha ,$$

where m_α is the moment of the data corresponding the monomial x^α:

$$m_\alpha = \tfrac{1}{q} \sum_{i=1}^{q} (p_i)^\alpha ,$$

remembering that $\alpha = (\alpha_0, \cdots, \alpha_n)$ and $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ here, because $x_0 = 1$.

In this way we obtain our desired parameterization

$$\phi(\psi, x_1, \cdots, x_n) = \sum_{|\alpha|=d} Q_\alpha(\psi)(x^\alpha - m_\alpha) . \qquad (14)$$

It parameterizes a family of polynomials that have zero mean value on the data set, and, therefore, nonempty zero sets. The other good property of this parameterization is that it is still quadratic and homogeneous in the coefficients of ψ, and it can be evaluated very inexpensively, as we will see in the next section.

VIII. Experimental Results

We have only implemented fitting using the parameterized family described in Section VIII–A. We have conducted a range of experiments, either minimizing the mean algebraic distance between a point and a surface or using Taubin's method, which is based on a better approximation of the geometric distance, and fitted quartic and sextic curves and surfaces to contour and range data. The Levenberg–Marquardt algorithm requires a set of initial parameters. In the experiments using the algebraic distance, we have either set all initial parameters to 1 or use as initial guess the quartic $x^4 + y^4 + z^4 = 1$, suitably translated and scaled to enclose all data points. Both sets of initial parameters have in general led to identical results at convergence. In the experiments based on Taubin's fitting algorithm, we used the generalized eigenvalue fit method for bounded curves and surfaces described in section IV–C for initialization.

We first present results obtained for algebraic surfaces whose defining polynomial $f = \psi + g$ has a positive definite leading form ψ. As shown in the Appendix, Lemma 5 actually shows that such a surface is enclosed in a sphere of radius $\max\{1, \mu/\nu\}$, where μ is some constant and ν is the minimum eigenvalue of the matrix $\Psi^2_{[2,2]}$. In practice, this means that the eigenvalues of $\Psi^2_{[2,2]}$ give us an indication of the volume enclosed by the surface, which allows us not only to ensure that a surface fit is bounded but also to control the size of the enclosed volume. This proves particularly valuable in avoiding large surface components that might appear where there are gaps in the data. We have implemented this idea in our fitting algorithm based on the algebraic distance: Since all the eigenvalues of $\Psi^2_{[2,2]}$ are positive, dividing the error of fit by the trace of this matrix has the effect of maximizing ν, therefore minimizing the enclosed volume. We have not used this idea in our implementation of Taubin's fitting algorithm yet.

Figs. 4 and 5 show experiments in curve fitting using the algebraic distance. The data points used in Fig. 4(a) actually belong to a quartic curve. As shown in the figure, the curve is correctly recovered. Fig. 4(b) shows the curve fitted to a polygon entered by hand. Again, a good approximation of the data is obtained. More interestingly, Fig. 5 shows another example with a large gap among the data points (once again entered by hand). In Fig. 5(a), the result of fitting an unconstrained quartic curve is shown: A large component is obtained because of the gap in the data. Fig. 5(b) shows the excellent approximation obtained when fitting a bounded, "minimum-area" quartic curve.

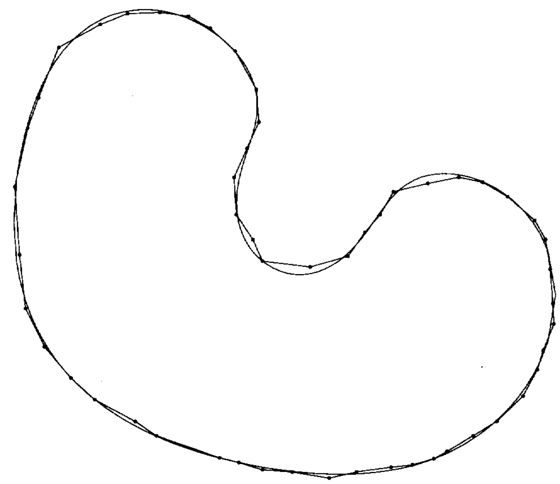


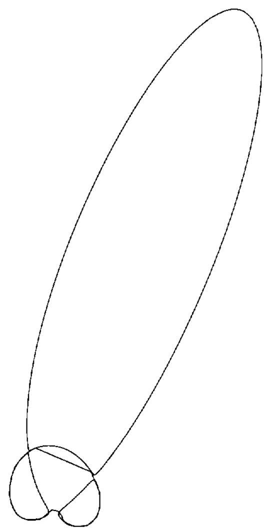
Fig. 4. Fitting quartic curves to 2-D points (drawn as small circles).

Figs. 6 and 7 show experiments in fitting bounded quartic surfaces to real range data, and again using the same implementation as in the previous examples. Fig. 6(a) shows the data from a single range image of a torus, and Fig. 6(b) shows the fitted surface. A good approximation of the visible part of the torus is recovered, and the overall shape is definitely torus-like, but the fit is clearly not as good. This is not surprising, since real data is noisy and fitting cannot be expected to "invent" the hidden parts of a surface.

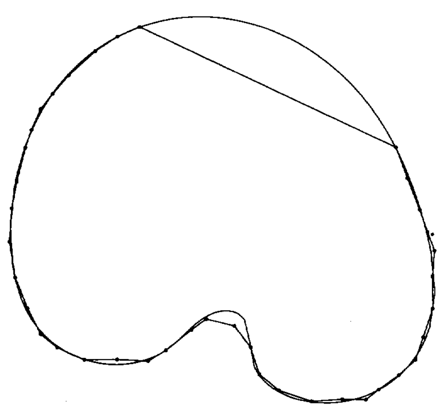
Fig. 7(a) shows a data set obtained by registering and merging three range images of a pepper. Again, despite noise and large gaps in the data, a reasonable surface model is recovered, as demonstrated by Fig. 7(b). The model is visually more satisfying than in the torus case, probably because the distribution of the data points over the pepper's surface is much more uniform.

Figs. 1, 8, 9, and 10 show examples of surface fitting based on Taubin's fitting algorithms. Figs. 1 and 8 further illustrate the effect of partial data on the resulting surfaces. Both data sets are subsets of real range data file from the NRCC range image database [32]. In both figures we show the data, the result of the unconstrained fourth-degree algebraic surface fit, the result of the generalized eigenvalue bounded fourth-degree fit (i.e., polynomials with leading form $(x_1^2 + x_2^2 + x_3^2)^2$ and linear in the rest of the coefficients), and the result of the general bounded fourth-degree fit.

Fig. 9 shows an example of general bounded fourth-degree surface fit to a data set without missing data. This data is a subset of a collection of CT scans slices. The surface is shown from two points of view, and both with and without the data superimposed. As can be observed, the surface gives a good global approximation of the shape, but the details are lost. The nose, the chin, and the ears are not well approximated by the surface fit. The same kind of behavior can be observed in Fig. 10. The ridges of the pepper are not well approximated by the fourth-degree surface fit.



(a)

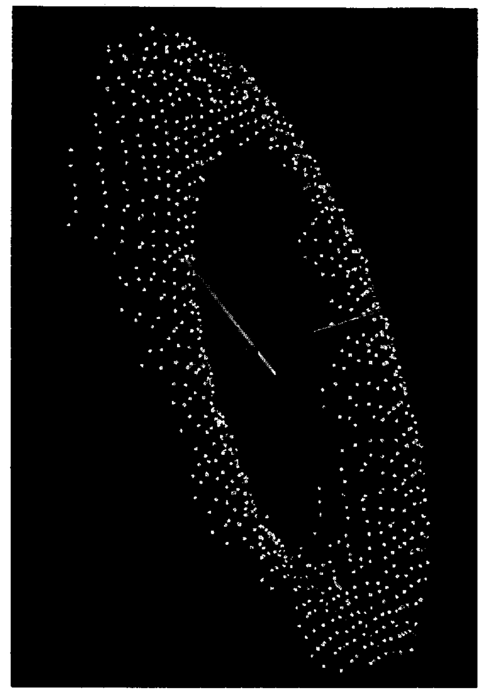


(b)

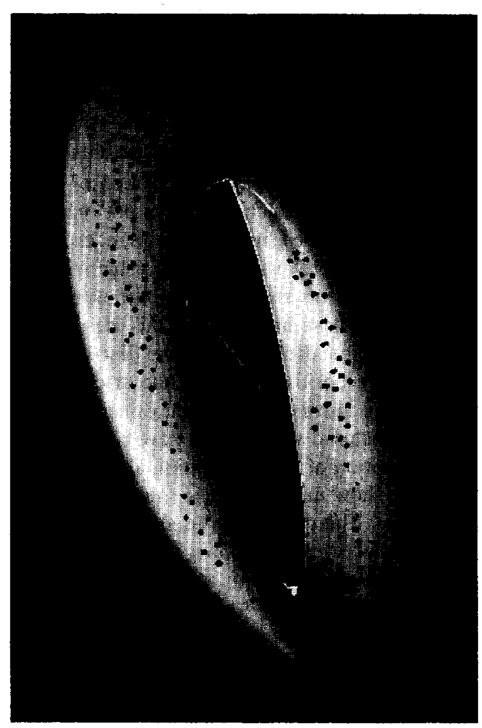
Fig. 5. Data points with a large gap among them. (a) Unconstrained fit. (b) Area-minimizing fit.

In an attempt to get those details, we increased the degree, and we fit a sixth-degree surface to the same data. The result, which can be seen in Fig. 10, is not satisfactory. The effect that we see here is basically the same as one can observe when fitting, say, a second-degree curve to data that can be well approximated by a straight line segment. Two branches of the curve will be close to a single branch of the data. In the case of surfaces the effect seems to be more severe. What seems to be the problem in this case is that the data can be coarsely approximated by an ellipsoid, and so a surface with parameters close to the union of three very similar ellipsoids will give a relatively good fit.

Experiments like this suggest that, apparently, the polynomials with coefficients close to reducible polynomials, i.e., polynomials that are products of lower degree polynomials (the sets of zeros of the product is the union of the sets

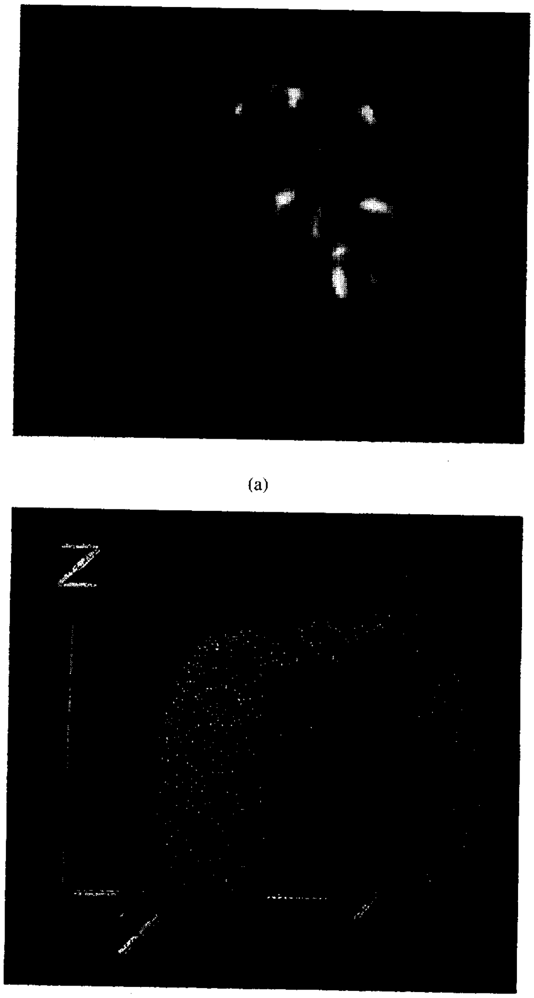


(a)



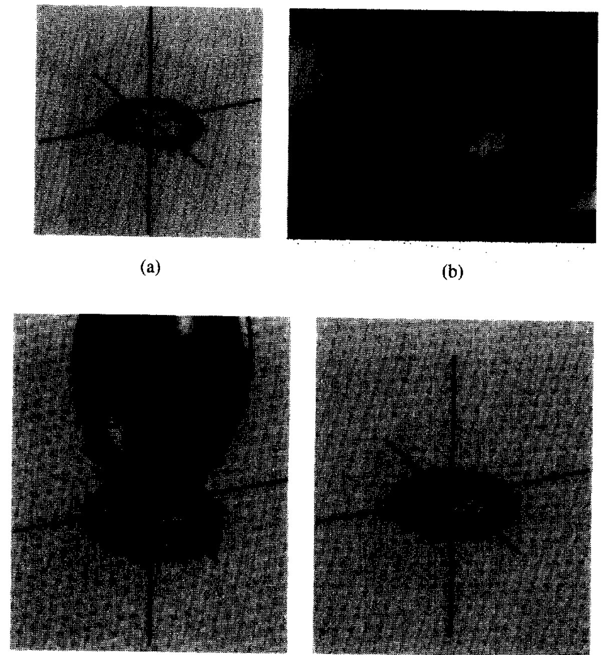
(b)

Fig. 6. (a) Real range data from a torus. (b) The recovered surface.



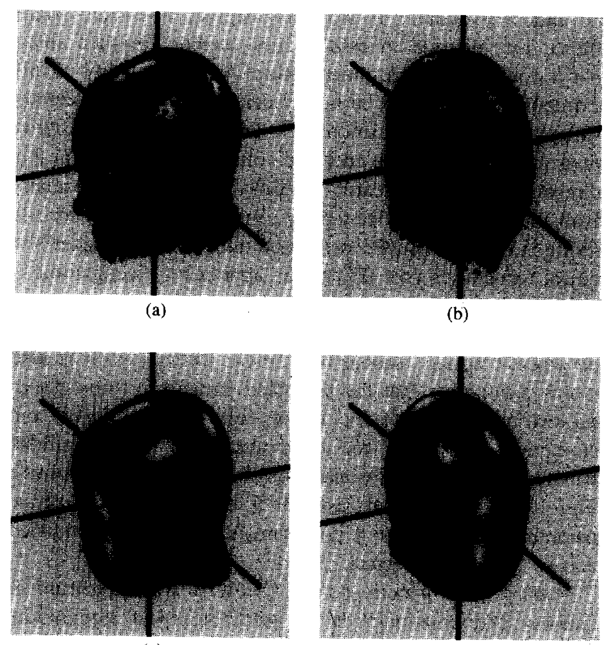
(a) (b)

Fig. 7. (a) Three range images of a pepper merged together as a single data set. (b) The surface fitted to the pepper data.



(a) (b) (c) (d)

Fig. 8. Region of file CHER1 from NRCC range database. (a) Data points represented in 3-D. (b) Unconstrained fourth-degree algebraic surface fit. (c) Generalized eigenvalue bounded fourth-degree fit. (d) General bounded fourth-degree fit.



(a) (b) (c) (d)

Fig. 9. Fourth-degree bounded algebraic surface fit to CT data. Two different views, with and without superimposed data.

of zeros of the factors), seem to be unstable regions of the parameter space. More research must be done to understand this phenomenon, and to control it. We intend to do so in the near future.

IX. Conclusion

We have described a technique for stabilizing the implicit function fitting process. The key drawback of implicit function fitting methods described in the literature thus far has been the unboundedness of the fitted curves and surfaces, or the lack of shape description power. In this paper, trying to solve these two problems, we have introduced two parameterized families of polynomials whose zero sets are always bounded, with enough flexibility in terms of shape description. Preliminary experimental results with 2-D curves and 3-D surfaces are encouraging, but they also point out some limitations that require further study.

First, it is clear that the most satisfying fits are obtained when the data covers most of the surface of the modeled object. As remarked before, this is not surprising, since fitting

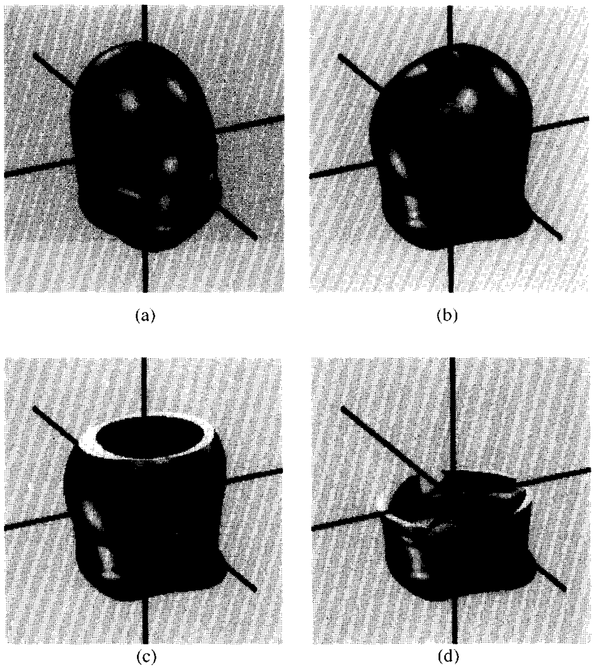


Fig. 10. Sixth-degree bounded algebraic surface fit to the same CT data of Fig. 9. The surface is shown from two different points of view, and with two sections to show the multiple branches.

cannot be expected to "invent" the shape of an object where no data is available. To a certain degree, algebraic surface fitting does provide an extrapolation method for such areas, but it does not provide much control over the quality of this extrapolation. It would be highly desirable to impose certain shape constraints, (e.g., smoothness or symmetry) on the fitting process. Minimizing the enclosed volume is a first step in this direction, but more research is clearly needed in this domain. It should be noted that we mostly plan to use algebraic surface fitting in off-line model construction rather than during on-line object recognition (see [27] for an approach to recognition of objects modeled by algebraic surfaces from monocular image contours). In this case, acquiring range data over most of an object's surface should not present difficulties. It is also worth mentioning that the presented parameterization places no restriction on the genus of the corresponding curves and surfaces, so some of the fitted surfaces have undesirable handles (holes) or additional components. (See for example Fig. 10.) Again, further research is required to characterize parameterized families of polynomials that are both bounded and of a particular genus.

From a more general point of view, a wide spectrum of surface representations can be used for object recognition. The number of shape parameters varies from very low (say, three for quadric surfaces), to intermediate (five to eight for superquadrics), and to very high (hundreds for polyhedra with many faces or deformable surfaces). Surfaces with few shape parameters are convenient for database storage and indexing, but these provide only rough shape descriptions, while surfaces with many shape parameters do not yield efficient indexing mechanisms but capture very fine shape detail. We believe that the niche of algebraic surfaces is somewhere between superquadrics and deformable surfaces. To be sure, superquadrics are always bounded and have genus zero, and, like generalized cylinders, they have symmetries that allow them to extrapolate data to the hidden side of an object during fitting. On the other hand, the geometry of algebraic surfaces is well understood from algebraic geometry, and since they do not have preferred axes of symmetry, they do not force structure onto objects that do not possess it and can be used to model symmetrical objects. They also afford many more degrees of freedom for shape description. As demonstrated by the pepper and face examples, this is still not sufficient for capturing very fine surface detail, which is not surprising. We believe that single-surface models afford enough detail for recognizing objects that are quite different (say, a pepper and a face), but not two objects that are quite close (say, two different faces). We plan to investigate part decomposition to remedy this problem in the future. We also plan to address the problem of directly constructing algebraic surface models from sets of video images (see [39] for preliminary results). Beyond modeling, we plan to further explore the application of algebraic surfaces to object recognition and indexing.

Appendix

The lexicographical order of multiindexes is defined as follows. If α and β are two multiindexes of the same size, we say that α precedes β and write $\alpha < \beta$, if for the first index k such that α_k differs from β_k we have $\alpha_k > \beta_k$. For example, for multiindexes of size 2 in three variables, the lexicographical order is

$$(2,0,0) < (1,1,0) < (1,0,1) < \\ (0,2,0) < (0,1,1) < (0,0,2) \,.$$

If α and β are multiindexes of different sizes, and the size of α is less than the size of β, we also say that α precedes β, and write $\alpha < \beta$.

Proof (Lemma 1): Apply the multinomial formula

$$(z_1 + \cdots + z_n)^d = \sum_{|\alpha|=d} \frac{d!}{\alpha!} z^\alpha$$

to the case $z_1 = x_1 y_1, \cdots, z_n = x_n y_n$ to obtain

$$\frac{1}{d!}(x^t y)^d = \sum_{|\alpha|=d} \frac{1}{\alpha!} x^\alpha y^\alpha = X_{[d]}(x)^t X_{[d]}(y) \,.$$

Proof (Lemma 2): Let ψ be a form of odd degree d that is not identically zero, i.e., such that $\psi(x) \neq 0$ for a certain point $x \in \mathbb{R}^n$. By homogeneity it follows that $\psi(-x) = \psi((-1)x) = (-1)^d \psi(x) = -\psi(x)$, because d is odd. Thjerefore, ψ cannot be definite.

Proof (Lemma 4): Let ψ be a nonpositive definite form of degree d, which is not identically zero. Due to homogeneity, the set of level zero of ψ is clearly unbounded. Let $\lambda > 0$. If ψ is nonnegative definite, its set of level λ is empty. Otherwise, let p_1 and p_2 be two nonzero points such that $\psi(p_1) = 0$ and $\psi(p_2) > 0$. By homogeneity, we can assume that both points are unit length. Now, for every $\theta > (\lambda/\psi(p_2))^{1/d}$, the form

$\psi(x)$ restricted to the sphere of radius θ attains a value larger than λ at the point θp_2 and the value zero at the point θp_1. By continuity of $\psi(x)$, there must exist a point p_3 on the sphere of radius θ such that $\psi(p_3) = \lambda$.

Proof (Lemma 5): Let $f = \psi + g$ be a polynomial of degree d, where ψ is a positive definite form of degree d and g is a polynomial of degree $< d$. We first show that there exists $\mu > 0$ such that $|g(x)| < \mu\|x\|^{d-1}$ for $\|x\| \geq 1$. For this g is written as a sum of forms of different degrees (11):

$$g(x) = \sum_{k=0}^{d-1} g_k(x) .$$

Clearly, $\mu \geq |g_0|$ is necessary. For $k > 0$ and $\|x\| \geq 1$, we write $g_k(x)$ in vector form (13) and we apply the Cauchy–Schwarz inequality, and then Lemma 1

$$\begin{array}{rcccl} |g_k(x)| & = & |G_{[k]}^t X_{[k]}(x)| & \leq & \|G_{[k]}\| \|X_{[k]}(x)\| \\ & = & \|G_{[k]}\| \dfrac{\|x\|^k}{\sqrt{k!}} & = & \dfrac{\|G_{[k]}\|}{\sqrt{k!}} \|x\|^{d-1} , \end{array}$$

with the last inequality because $\|x\| \geq 1$. It is sufficient to take

$$\mu = |g_0| + \sum_{k=1}^{d-1} \frac{\|G_{[k]}^t\|}{\sqrt{k!}} \tag{15}$$

to obtain the desired result. Now we prove that the set $\{x : f(x) = \lambda\}$ is bounded, for every $\lambda \in \mathbb{R}$. By including λ as part of the independent term of the polynomial g, it is sufficient to consider the case $\lambda = 0$. Since $\psi(x) \geq \psi_{\text{MIN}} \|x\|^d$, for $\|x\| \geq 1$ we have

$$\begin{array}{rcl} 0 & = & f(x) \\ & = & \psi(x) + g(x) \\ & \geq & \psi_{\text{MIN}} \|x\|^d - \mu \|x\|^{d-1} \\ & \geq & \psi_{\text{MIN}} \|x\| - \mu , \end{array}$$

or, equivalently,

$$\|x\| \leq \frac{\mu}{\psi_{\text{MIN}}} .$$

That is, if $f(x) = 0$, then $\|x\| \leq \max\{1, \mu/\psi_{\text{MIN}}\}$.

Proof (Lemma 6): Let $f = \psi + g$ be a polynomial of degree d, where ψ is an indefinite form of degree d and g is a polynomial of degree $< d$. If μ is the constant (15), for $\|x\| \geq 1$ we have

$$f(x) \geq \psi(x) - |g(x)| \geq \psi(x) - \mu\|x\|^{d-1}$$

and

$$f(x) \leq \psi(x) + |g(x)| \leq \psi(x) + \mu\|x\|^{d-1} .$$

Now, since ψ is indefinite, there exist two points p_1 and p_2 such that $\psi(p_1) > 0$ and $\psi(p_2) < 0$. By the homogeneity of ψ, we can also choose them of unit length $\|p_1\| = \|p_2\| = 1$. If we apply the first inequality for $x = \theta p_1$ and $\theta = \|x\| \geq 1$, we obtain

$$f(\theta p_1) \geq \psi(\theta p_1) - \mu\|\theta p_1\|^{d-1} = \theta^{d-1}(\theta\psi(p_1) - \mu) ,$$

which is positive for $\theta > \mu/\psi(p_1)$. If we apply the second inequality for $x = \theta p_2$ and $\theta = \|x\| \geq 1$, we obtain

$$\begin{array}{rcll} f(\theta p_2) & \leq & \psi(\theta p_2) + \mu\|\theta p_2\|^{d-1} & \\ & = & \theta^{d-1}(\theta\psi(p_2) + \mu) & < 0 , \end{array}$$

which is negative for $\theta > -\mu/\psi(p_2)$. Now, for $\theta > \max\{\mu/\psi(p_1), -\mu/\psi(p_2)\}$, the polynomial $f(x)$ restricted to the sphere of radius θ attains a positive value at one point and a negative value at another point. By continuity of $f(x)$, there must exist a point p_3 on the sphere of radius θ such that $f(p_3) = 0$.

Proof (Lemma 7): First of all, note that the two forms are linearly dependent if and only if the determinant

$$\begin{vmatrix} \psi_1(x) & \psi_2(x) \\ \psi_1(y) & \psi_2(y) \end{vmatrix}$$

is identically zero as a polynomial in two sets of variables, $x = (x_1, \cdots, x_n)^t$ and $y = (y_1, \cdots, y_n)^t$. In particular, if the two forms are linearly independent, there exist two points p_1 and p_2 such that the determinant

$$\begin{vmatrix} \psi_1(p_1) & \psi_2(p_1) \\ \psi_1(p_2) & \psi_2(p_2) \end{vmatrix} \neq 0 .$$

In this case, the linear system of two equations in μ_1 and μ_2,

$$\begin{cases} \mu_1\psi_1(p_1) + \mu_2\psi_2(p_1) & = 1 \\ \mu_1\psi_1(p_2) + \mu_2\psi_2(p_2) & = -1 \end{cases}$$

has a unique solution, and the form $\mu_1\psi_1(x) + \mu_2\psi_2(x)$ is indefinite.

Proof (Lemma 8): Let p_1 and p_2 be two different points on the unit sphere, and let A be an orthogonal matrix such that $Ap_1 = p_2$ (for example, the Householder transformation $A = I - 2vv^t/\|v\|^2$, with $v = p_1 - p_2$ [21]). Since ψ is invariant under orthogonal coordinate transformations, we have $\psi(p_2) = \psi(Ap_1) = \psi(p_1)$. Since this is true for an arbitrary pair of points, ψ is constant on the unit sphere, in particular $\psi_{\text{MAX}} = \psi_{\text{MIN}}$, and so (from (14)) $\psi(x) = \psi_{\text{MIN}}\|x\|^d$ for every $x \in \mathbb{R}^n$. If d is also odd, we also have $\psi(-x) = -\psi(x)$ for every point, and so $\psi(x) \equiv 0$.

Proof (Lemma 9): If we write the forms ξ_i of (15) in vector form $\xi_i(x) = \Xi_{[k]i}^t X_{[k]}(x)$, we can also rewrite ψ as a quadratic form in the vector of monomials $X_{[k]}(x)$:

$$\psi = X_{[k]}^t \Big(\sum_{i=1}^{h} \Xi_{[k]i} \Xi_{[k]i}^t \Big) X_{[k]} = X_{[k]}(x)^t M X_{[k]}$$

where by construction M is a nonnegative definite symmetric $h_k \times h_k$ matrix. Let N be a square root of M, i.e., any $h_k \times h_k$ matrix such that $NN^t = M$. For example, we can take it lower triangular, using the Cholesky factorization of M. Now we go backward in the previous equation. If $\Lambda_1, \cdots, \Lambda_{h_k}$ are the columns of N and $\lambda_1(x) = \Lambda_1^t X_{[k]}(x), \cdots, \lambda_{h_k}(x) = \Lambda_{h_k}^t X_{[k]}(x)$, we have

$$\begin{array}{rcl} \psi & = & X_{[k]}^t M X_{[k]} \\ & = & \left(N^t X_{[k]}\right)^t \left(N^t X_{[k]}\right) \\ & = & \displaystyle\sum_{i=1}^{h_k} \lambda_i(x)^2 . \end{array}$$

For Lemma 10 we need to introduce some notation and prove a few preliminary results. Let $D = (\partial/\partial x_1, \cdots, \partial/\partial x_n)^t$ be the vector of first-order partial derivatives. For every form ψ of degree d there is a corresponding homogeneous linear differential operator

$$\psi(D) = \sum_{|\eta|=d} \tfrac{1}{\eta!}\Psi_\eta D^\eta = \Psi_{[d]}^t D_{[k]} ,$$

where $D_{[d]} = X_{[k]}(D)$. Every homogeneous linear differential operator of degree d can be written in this form in a unique way, i.e., the vector space of linear differential operators of order d is a vector space of the same dimension h_d, and the map $\psi(x) \mapsto \psi(D)$ defines an isomorphism of vector spaces. In fact, this is the dual space of the vector space of forms of degree d, with respect to the inner product

$$\begin{aligned} \langle \psi, \psi \rangle &= \psi(D)\psi \\ &= \psi(D)\psi \\ &= \sum_{|\eta|=d} \tfrac{1}{\eta!}\Psi_\eta \Psi_\eta \\ &= \Psi_{[d]}^t \Psi_{[d]} . \end{aligned}$$

This inner product is invariant with respect to the action of the orthogonal group $\mathcal{O}(n)$ onto the space of forms of degree d (see also [24], [42]).

If ψ is a form of degree d with coefficients $\{\Psi_\alpha : |\alpha| = d\}$, since $\Psi_\alpha = D^\alpha \psi$ for every multiindex α, by construction we have $\Psi_{[d]} = D_{[d]}\psi$. Also, for each pair of nonnegative integers j and k, such that $d = j + k$, we have $\Psi_{[j,k]} = D_{[j,k]}\psi$, where $D_{[j,k]} = X_{[j,k]}(D)$, and $\Psi_{[j,k]}$ is the $h_j \times h_k$ matrix

$$\Psi_{[j,k]} = \left\{ \sqrt{\tfrac{1}{\alpha!\,\beta!}}\ \Psi_{\alpha+\beta} : |\alpha| = j, |\beta| = k \right\} .$$

Furthermore,

Lemma 14: If ψ is a form of degree d and (j,k) is a pair of nonnegative integers such that $j + k = d$, then $D_{[j]}\psi(x) = \Psi_{[j,k]}X_{[k]}(x)$. Equivalently, for every multiindex α of size j, we have

$$D^\alpha \psi = \sum_{|\beta|=k} \tfrac{1}{\beta!}\Psi_{\alpha+\beta} x^\beta .$$

Proof: For each multiindex η of size j, there is a one-to-one correspondence between the following two sets:

$$\{\alpha : |\alpha| = d, \alpha \geq \eta\} = \{\beta + \eta : |\beta| = k\} ,$$

with $\alpha \geq \eta$ meaning $\alpha_1 \geq \eta_1, \cdots, \alpha_n \geq \eta_n$. For every other multiindex α of size d, we have

$$D^\eta(x^\alpha) = \begin{cases} \frac{\alpha!}{\beta!}x^\beta & \text{if } \alpha \geq \eta \text{ and } \alpha = \beta + \eta \\ 0 & \text{otherwise} \end{cases} ,$$

so that

$$D^\eta \psi(x) = \sum_{|\alpha|=d} \tfrac{1}{\alpha!}\Psi_\alpha D^\eta(x^\alpha) = \sum_{|\beta|=k} \tfrac{1}{\beta!}\Psi_{\beta+\eta} x^\beta ,$$

which, if written in matrix form, is the desired result.

Now we can prove Euler's theorem.

Proof (Lemma 10): Since the form ψ is homogeneous of degree d, we have the identity $\theta^d \psi(x) \equiv \psi(\theta x)$ in $n + 1$ variables $\theta, x_1, \cdots, x_n$. Differentiating j times with respect to θ, and using the chain rule, we obtain

$$\begin{aligned} \tbinom{d}{j}\theta^k \psi(x) &= \sum_{|\eta|=j} \tfrac{1}{\eta!}D^\eta \psi(\theta x) x^\eta \\ &= X_{[j]}^t(x) D_{[j]}\psi(\theta x) . \end{aligned}$$

Finally, we evaluate the previous expression at $\theta = 1$ and substitute the vector of partial derivatives according to Lemma 14.

Proof (Lemma 11): Since $\Psi_{[k,k]}$ is symmetric and nonsingular, $\Psi_{[k,k]}^2$ is symmetric and positive definite, i.e., $Y^t \Psi_{[k,k]}^2 Y > 0$ for every nonzero vector Y. In particular,

$$Q(\psi)(x) = X_{[k]}^t(x)\Psi_{[k,k]}^2 X_{[k]}(x) > 0$$

for every $x \neq 0$.

Proof (Corollary 1): Lemma 11 and Lemma 5.

Now, we can rewrite the quadratic map Q in three equivalent ways:

$$\begin{aligned} Q(\psi) &= \sum_{|\alpha|=k} \tfrac{1}{\alpha!}(D^\alpha \psi)^2 \\ &= \|D_{[k]}\psi\|^2 \\ &= X_{[k]}^t \Psi_{[k,k]}^2 X_{[k]} . \end{aligned} \tag{16}$$

In order to establish the covariance property of the map Q (Lemma 12), we first need to study the transformation rules of vectors of monomials and vectors of coefficients of forms. If $x' = Ax$ is a nonsingular linear transformation, for every form ψ, the polynomial $\psi(Ax)$ is a form of the same degree. In particular, every component of the vector $X_{[d]}(Ax)$ can be written in a *unique way* as a linear combination of the elements of $X_{[d]}$, or in matrix form, i.e.,

$$X_{[d]}(Ax) = A_{[d]}X_{[d]}(x) ,$$

where $A_{[d]}$ is a nonsingular $h_d \times h_d$ matrix. The map $A \mapsto A_{[d]}$ is a *dth degree polynomial representation* of the group $\mathrm{GL}(n)$, and the matrix $A_{[d]}$ is a *dth degree representation matrix* of A. Furthermore,

Lemma 15: The map $A \mapsto A_{[d]}$ has the following properties.

1) It defines a *faithful linear representation*, a one-to-one homomorphism of groups, of the group of nonsingular $n \times n$ matrixes $\mathrm{GL}(n)$ into the group of nonsingular $h_d \times h_d$ matrixes $\mathrm{GL}(h_d)$, i.e., for every pair of nonsingular matrixes A and B, we have
   a) The map preserves products $(AB)_{[d]} = A_{[d]}B_{[d]}$.
   b) The map is one to one: If $A_{[d]} = B_{[d]}$, then $A = B$.
   c) The matrix $A_{[d]}$ is nonsingular, and $(A_{[d]})^{-1} = (A^{-1})_{[d]}$.

2) It preserves transposition, i.e., for every nonsingular matrix A, we have $(A^t)_{[d]} = (A_{[d]})^t$. In particular, if A is symmetric, positive definite, or orthogonal, so is $A_{[d]}$.
3) If A is lower triangular, so is $A_{[d]}$. In particular, if A is diagonal, so is $A_{[d]}$.
4) The determinant of $A_{[d]}$ is equal to $|A|^m$, with $m = \binom{n+d-1}{n}$.

Proof: This is a well-known result in the theory of representations of Lie groups [10]. For an elementary proof see [38] or [44].

These properties of the representation matrixes play a central role with respect to the transformation rules of coefficients of forms.

Lemma 16: If $\psi(x)$ is a form of degree d and $\psi'(x') = \psi(A^{-1}x')$, then $\Psi'_{[d]} = A_{[d]}^{-t}\Psi_{[d]}$. In particular, if A is orthogonal, then $\Psi'_{[d]} = A_{[d]}\Psi_{[d]}$. Also, for each pair of nonnegative integers j and k such that $j+k=d$, we have $\Psi'_{[j,k]} = A_{[j]}^{-t}\Psi_{[j,k]}A_{[k]}^{-1}$. In particular, if A is orthogonal, then $\Psi'_{[j,k]} = A_{[j]}\Psi_{[j,k]}A_{[k]}^t$.

Proof: If $x' = Ax$ is a coordinate transformation, the partial derivatives are related by the chain rule

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} = \sum_{j=1}^{n} A_{ji}\frac{\partial}{\partial x'_j} ,$$

or, in matrix form, $D' = (\partial/\partial x'_1, \cdots, \partial/\partial x'_n)^t = A^{-t}D$. Then,

$$\begin{aligned} D'_{[k]} &= X_{[k]}(D') \\ &= X_{[k]}(A^{-t}D) \\ &= (A^{-t})_{[k]}X_{[k]}(D) \\ &= A_{[k]}^{-t}D_{[k]} , \end{aligned} \tag{17}$$

and

$$\begin{aligned} D'_{[k,j]} &= X_{[k]}(D')X_{[j]}^t(D') \\ &= X_{[k]}(A^{-t}D)X_{[j]}^t(A^{-t}D) \\ &= A_{[k]}^{-t}X_{[k]}(D)X_{[j]}^t(D)A_{[j]}^{-1} \\ &= A_{[k]}^{-t}D_{[k,j]}A_{[j]}^{-1} . \end{aligned} \tag{18}$$

Finally, we have, from (17) and Lemma 14,

$$\Psi'_{[d]} = D'_{[d]}\psi'(x') = A_{[d]}^{-t}D_{[d]}\psi(x) = A_{[d]}^{-t}\Psi_{[d]} ,$$

and if $d = j+k$, from (23),

$$\begin{aligned} \Psi'_{[j,k]} &= D'_{[j,k]}\psi'(x') \\ &= A_{[j]}^{-t}\big(D_{[j,k]}\psi(x)\big)A_{[k]}^{-1} \\ &= A_{[j]}^{-t}\Psi_{[j,k]}A_{[k]}^{-1} . \end{aligned}$$

The first immediate consequence of these transformation rules is that

Corollary: The representation of a form given by Euler's theorem is invariant under homogeneous linear transformations, i.e., with the same hypotheses of the previous lemma,

$$X_{[j]}^t(x')\Psi'_{[j,k]}X_{[k]}(x') = X_{[j]}^t(x)\Psi_{[j,k]}X_{[k]}(x) .$$

Proof: Immediate from Lemma 16.

Finally, we can prove Lemma 12.

Proof (Lemma 12): With the same hypotheses of Lemma 16, apply Lemma 16 with $j = k$ and then use equation 21 in both members:

$$\begin{aligned} Q(\psi')(x') &= X_{[k]}^t(x')\Big(\Psi'_{[k,k]}\Big)^2 X_{[k]}(x') \\ &= X_{[k]}^t(x)\Psi_{[k,k]}^2X_{[k]}(x) \\ &= Q(\psi)(x) . \end{aligned}$$

We will now prove Lemma 13. Looking at the quadratic map Q as a smooth function of the coefficients of a form ψ of degree d, the differential of Q evaluated at ψ is a linear map

$$dQ(\psi) : \mathbb{R}^{h_d} \to \mathbb{R}^{h_d} ,$$

and Q is one to one in a neighborhood of ψ if and only if $dQ(\psi)$ is nonsingular. Let M_ψ denote the matrix of $dQ(\psi)$ in the basis associated with the vector of monomials $X_{[d]}$. That is, if $\theta = \Theta_{[d]}^tX_{[d]}$ and $\xi = \Xi_{[d]}^tX_{[d]}$ are two forms of degree d such that $dQ(\psi)(\theta) = \xi$, then $\Xi_{[d]} = M_\psi\Theta_{[d]}$. M_ψ is an $h_d\times h_d$ matrix whose components are homogeneous linear functions of the coefficients of ψ, the partial derivatives of the quadratic functions $Q_\alpha(\psi)$ with respect to the coefficients Ψ_β of ψ. Furthermore, $dQ(\psi)$ is nonsingular if and only if the determinant $|M_\psi|$ is nonzero. Since $|M_\psi|$ is an homogeneous polynomial of degree h_d in the coefficients of ψ, for the proof of Lemma 13 it is sufficient to show that $|M_\psi|$ is not identically zero because in such a case, the set of forms $\{\psi \in \mathbb{R}^{h_d} : |M_\psi| = 0\}$ would be a variety of codimension one, and for every form ψ not in this set the differential $dQ(\psi)$ would be nonsingular. Now, in order to prove that $|M_\psi|$ is not identically zero, we have only to show a particular form ψ such that $|M_\psi| \neq 0$. The next two lemmas show how to compute $dQ(\psi)$ and M_ψ, and the third one shows how to construct a form ψ such that $|M_\psi| \neq 0$, completing the proof of Lemma 13.

Lemma 17: Let ψ and ξ be forms of degree $d = 2k$. Then,

$$\begin{aligned} &dQ(\psi)(\xi)(x) \\ &\quad = 2\sum_{|\alpha|=|\beta|=|\gamma|=k}\sum\sum \tfrac{1}{\alpha!\beta!\gamma!}\Psi_{\alpha+\gamma}\Xi_{\alpha+\beta}x^{\beta+\gamma} \\ &\quad = 2\mathrm{trace}\big(\Psi_{[k,k]}\Xi_{[k,k]}X_{[k,k]}(x)\big) . \end{aligned}$$

Proof: We first apply Lemma 14 to the expansion of Q in (16) in order to obtain

$$\begin{aligned} Q(\psi) &= \sum_{|\alpha|=k} \tfrac{1}{\alpha!}(D^\alpha\psi)^2 \\ &= \sum_{|\alpha|=|\beta|=|\gamma|=k}\sum\sum \tfrac{1}{\alpha!\beta!\gamma}\Psi_{\alpha+\beta}\Psi_{\alpha+\gamma}x^{\beta+\gamma} . \end{aligned}$$

Then, the differential $dQ(\psi)(\xi)$ is the coefficient of the linear term of $F(\psi + t\xi)$ with respect to t

$$\begin{aligned} &\sum_{|\alpha|=\beta|=|\gamma|=k}\sum\sum \tfrac{1}{\alpha!\beta!\gamma}(\Psi_{\alpha+\beta}\Xi_{\alpha+\gamma} + \Xi_{\alpha+\beta}\Psi_{\alpha+\gamma})x^{\beta+\gamma} \\ &\quad = 2\sum_{|\alpha|=|\beta|=|\gamma|=k}\sum\sum \tfrac{1}{\alpha!\beta!\gamma}\Psi_{\alpha+\beta}\Xi_{\alpha+\gamma}x^{\beta+\gamma} . \end{aligned}$$

Lemma 18: Let ψ be a form of degree $d = 2k$, and let

$$G_\psi(x,y) = 2\text{trace}\big(X_{[k,k]}(y)\Psi_{[k,k]}X_{[k,k]}(x)\big) .$$

If μ and ν are multiindexes of size $d = 2k$, the (μ,ν)th element of the matrix M_ψ is

$$M_{\psi(\mu,\nu)} = D_x^\mu D_y^\nu G_\psi(x,y) .$$

In particular, M_ψ is symmetric.

Proof: Let's consider the form $\psi_y(x) = (y^t x)^d/d!$, where $y \in \mathbb{R}^n$. In this case, $\Psi_{[k,k]} = X_{[k,k]}(y)^t$. Consider the vector y as a new vector of indeterminates, and we have $G_\psi(x,y) = dQ(\psi)(\psi_y)(x)$. If μ and ν are multiindexes of size $d = 2k$, the (μ,ν)th element of the matrix M_ψ can be obtained by differentiating G_ψ

$$M_{\psi(\mu,\nu)} = D_x^\mu D_{\psi_\nu} dQ(\psi)(\psi)(x) = D_x^\mu D_y^\nu G_\psi(x,y) .$$

Finally, since $\Psi_{[k,k]}$ is symmetric, so is G_ψ, and based on the last equation, so is M_ψ.

Lemma 19: Let ψ be a form of degree $d = 2k$. Then,

1) If $\Psi_{[k,k]}$ is positive definite, so is M_ψ.
2) There exists a form ψ of degree $d = 2k$ such that $\Psi_{[k,k]}$ is positive definite.

Proof: 1) The matrix M_ψ defines a quadratic form on $\mathbb{R}^{h_d}$

$$Q_\psi(\xi) = \Xi_{[d]}^t M_\psi \Xi_{[d]} ,$$

where $\xi = \Xi_{[d]}^t X_{[d]}$ is another form of degree d. The matrix M_ψ is positive definite if and only if the quadratic form Q_ψ is positive definite, and it is not difficult to see that Q_ψ can also be written in the following way:

$$\begin{aligned} Q_\psi(\xi) &= \text{trace}\big(\Xi_{[k,k]}\Psi_{[k,k]}\Xi_{[k,k]}\big) \\ &= \sum_{i=1}^{h_k} u_i^t \Psi_{[k,k]} u_i , \end{aligned}$$

where $u_1, \cdots, u_{h_k}$ are the columns of the matrix $\Xi_{[k,k]}$. If ξ is a nonzero form, at least one of these vectors is nonzero, and since $\Psi_{[k,k]}$ is positive definite, the sum on the right side is positive.

2) Let μ be a measure of compact support on $\mathbb{R}^n$. Furthermore, let us require μ to have nonempty interior, i.e., its support contains an open set. The moments of degree $d = 2k$ of μ define a form ψ with coefficient vector

$$\Psi_{[d]} = \int X_{[d]}(x)d\mu(x) .$$

In particular, the matrix of coefficients $\Psi_{[k,k]}$ can also be represented in integral form

$$\Psi_{[k,k]} = \int X_{[k,k]}(x)d\mu(x) ,$$

which shows that $\Psi_{[k,k]}$ is nonnegative definite. If $\Psi_{[k,k]}$ were singular, then for certain nonzero form ξ of degree k, with vector of coefficients $\Xi_{[k]}$, we must have

$$\begin{aligned} 0 &= \Xi_{[k]}^t \Psi_{[k,k]} \Xi_{[k]} \\ &= \int \left(\Xi_{[k]}^t X_{[k]}(x)\right)^2 d\mu(x) \\ &= \int \xi(x)^2 d\mu(x) , \end{aligned}$$

i.e., the form ξ must be identically zero on the support of the measure μ. However, since the support of μ contains an open set, this is possible only if ψ is identically zero, which is a contradiction. So, $\Psi_{[k,k]}$ is nonsingular.

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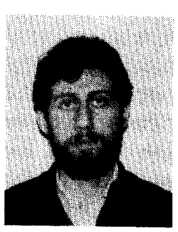

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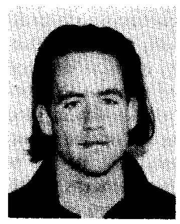

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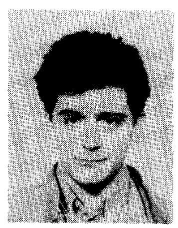

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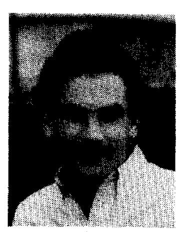

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