

# Nonplanar Curve and Surface Estimation in 3-Space

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**Abstract :** This paper addresses the problem of minimal parameter representation and estimation for complex planar and nonplanar curves, and surfaces. The representation is based on concepts from Algebraic Geometry : a surface is the set of roots of a polynomial of three variables, and a curve is the intersection of two different surfaces. We show that the surfaces of an interesting complex objects in 3-space can be represented by single high degree polynomials, and a similar statement applies to complex curves in 3-space. An approximate expression for the mean square distance from a set of points to a curve or surface is developed, not only for quadratic surfaces, but for surfaces and curves defined by polynomials of higher degree. Given a set of points, curve estimation corresponds to the minimization of the previous expression over all the pairs of polynomials of a given degree. Surface estimation is based on the same procedure. We present a computationally efficient algorithm to carry out the minimization without using nonlinear optimization techniques. The approximate mean square distance has the same meaning, not only for dimension three, but for any other space dimension, obtaining as a byproduct a method for curve estimation for 2D-vision. This work has immediate application to object or curve position estimation, stereo vision, etc. [4].

## I. INTRODUCTION

Object recognition and position estimation are two central issues in Computer vision. The selection of an internal representation for the objects the vision system has to deal with that is well adapted for the two objectives mentioned above, is the most important decision to take. In recent years, solid objects have been represented as sets of surface patches, usually planar. Lately quadric patches have been considered too [1]. Non planar curves arise naturally in this context, and at least the following three reasons justify our interest to represent and estimate nonplanar curves:

- 1) The intersection of two surface patches is, usually, a nonplanar curve.
- 2) If the object has patterns on the surfaces, the boundaries of the patterns are usually nonplanar curves.
- 3) For objects consisting of many small surface patches, estimating nonplanar curves of surface intersections may be more usefull than estimating the surface patches.

There is no reason to restrict the surfaces considered to be quadric surfaces. In general we will represent a surface as the set of roots of a polynomial in three variables, and a curve as the intersection of two different surfaces : the common roots of two linearly independent polynomials of three variables. In this way, an object will be represented, either as a set of surface patches, or as a set of surface patches specifying curve patches, or as both . Given a maximum degree for the polynomials considered, and given a set of data points in space assumed to belong to the same surface or curve patch, we want to estimate the coefficients of a polynomial surface or pair of polynomials for a curve which minimize the mean square distance to the data points for a surface or curve respectively. However, there is no closed form expression for the mean square distance from one of these curves or surfaces to the data set. In this paper we develop a first order approximation for the square of the distance from a point to a curve or surface. The mean value of this function, on a fixed set of data points, is a highly nonlinear function of the coefficients of the polynomials, and computationally costly to minimize; but a second approximation gives us a new expression, whose minimization does not involve nonlinear optimization techniques, and turns out to be an efficient procedure. And even more, the surface or curve fit resulting from the minimization process is invariant with respect to affine transformations of the data set, particularly with respect to rigid body transformations. The approximate mean square distance has the same

meaning, not only for 3D objects, but also for 2D objects and manifolds in spaces of dimension larger than three. In this way, this tool is usefull for 2D-vision and for abstract n-dimensional spaces too.

## II. APPROXIMATE SQUARE DISTANCE

The intuitive idea comes from the one dimensional case. Let  $y = f(x)$  be a function of only one variable. Let  $x$  be any point, such that  $f(x) \neq 0$ , and  $z$  be the root of  $f$  closest to  $x$ .

$$0 = f(z) \approx f(x) + f'(x)(z-x). \text{ Hence, } (x-z)^2 \approx \frac{[f(x)]^2}{[f'(x)]^2}$$

Note,  $(x-z)^2$  is the square distance from  $x$  to the set of roots of  $f$ . This argument can be generalized to the distance from a point to a curve in three space. Let  $C = \{ y : f_1(y) = f_2(y) = 0 \}$  be a curve. Then

$$\text{distance}(x, C)^2 \approx f(x)' \left[ \begin{matrix} -\frac{\partial f(x)}{\partial x} \\ -\frac{\partial f(x)}{\partial x} \end{matrix} \right]' f(x) \quad (1)$$

where  $f(x) = (f_1(x), f_2(x))'$ , and  $\frac{\partial f(x)}{\partial x}$  is the  $2 \times 3$  jacobian matrix.

The expression (1) is also true for any smooth function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^k$  such that  $k \leq n$  and  $\frac{\partial f(x)}{\partial x}$ , which is  $k \times n$ , has rank  $k$ .

The following are particular cases :

- 1) Planar curves in 2D-space :  $n = 2$  and  $k = 1$
- 2) Surfaces in 3D-space :  $n = 3$  and  $k = 1$
- 3) Nonplanar curves in 3D-space :  $n = 3$  and  $k = 2$

A surface is represented as the set of roots of a polynomial  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , and a curve is represented as the intersection of two surfaces. For example the intersection of two cylinders is the set of common roots of two polynomials :

$$f(x) = \begin{bmatrix} x^2 + (z-1)^2 - 4 \\ y^2 + (z+1)^2 - 4 \end{bmatrix} = \begin{bmatrix} -3 - 2z + z^2 + x^2 \\ -3 + 2z + z^2 + y^2 \end{bmatrix}$$

We can write  $f(x) = F X$ , where  $F$  is the matrix of coefficients

$$F = \begin{bmatrix} -3 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and  $X$  is the vector of monomials

$$X = (1 \ z \ y \ x \ z^2 \ zy \ zx \ y^2 \ yx \ x^2 \ )'$$

With this representation, every statement about  $f$  translates into a statement about the matrix  $F$ , particularly the approximate square distance :

$$\text{distance}(x, C)^2 \approx X' F \left[ F \begin{pmatrix} \frac{\partial X}{\partial x} \\ \frac{\partial X}{\partial x} \end{pmatrix}' F' \right]^{-1} F X \quad (2)$$

## III. APPROXIMATE MEAN SQUARE DISTANCE

Let  $P = \{ x_1, \dots, x_p \}$  be a set of data points, say range data; and let  $f(x) = F X$  be a polynomial. Based on the previous expression (2), the following function is an approximate mean square distance from  $P$  to  $C$  :

$$\Phi_P(F) = \frac{1}{p} \sum_{1 \leq i \leq p} \{ X_i' F \left[ F \begin{pmatrix} \frac{\partial X}{\partial x} \\ \frac{\partial X}{\partial x} \end{pmatrix}' F' \right]^{-1} F X_i \}$$

The problem is that  $\Phi_P(F)$  is costly to evaluate and difficult to minimize because of the summation. Our solution is to replace the variable matrix  $\left( \frac{\partial X}{\partial x} \right) \left( \frac{\partial X}{\partial x} \right)'$  with the constant matrix  $\frac{1}{p} \sum_{1 \leq i \leq p} \left( \frac{\partial X}{\partial x} \frac{\partial X}{\partial x} \right)'$ .

The new approximate mean square distance is denoted  $\xi_P(F)$ , and it measures the approximate mean square distance in the mean normal direction to the curve or surface.

#### IV. ESTIMATION = MINIMIZATION OF $\xi_p(F)$

Given the set of data points and a maximum degree for the polynomials considered, the process of curve estimation becomes the minimization of  $\xi_p(F)$ , over all the matrices  $F$  for which  $\xi_p(F)$  is well defined. If we denote

$$M = \frac{1}{p} \sum_{1 \leq i \leq p} XX' \quad \text{and} \quad N = \frac{1}{p} \sum_{1 \leq i \leq p} \left\{ \frac{\partial X}{\partial x} \frac{\partial X'}{\partial x} \right\}$$

and the matrix  $N$  is the identity matrix, we prove [4] that the rows of the matrix  $F$  which minimizes  $\xi_p(F)$  can be chosen as eigenvectors of  $M$  corresponding to the two smallest eigenvalues.

In general  $N$  is not the identity matrix, even worse, it is usually singular; but we prove that the general case can be reduced to the previous one [4]. The following steps define the algorithm:

- 1) Compute the diagonal decomposition of  $N : N = U' D U$ , with  $U U' = I$ ,  $D = \text{diag}(d_1, \dots, d_m)$ , and  $d_1 \geq \dots \geq d_s > d_{s+1} = \dots = d_m = 0$
- 2) If  $U_1, \dots, U_m$  denote the rows of  $U$ ,

$$\text{take } H = \begin{bmatrix} U_{s+1} \\ \dots \\ U_m \end{bmatrix} \quad \text{and} \quad G = \text{diag} \begin{bmatrix} 1/\sqrt{d_1} \\ \dots \\ 1/\sqrt{d_s} \end{bmatrix} \begin{bmatrix} U_1 \\ \dots \\ U_s \end{bmatrix}$$

- 3) Compute  $T = M H' (H M H')^{-1} H$
- 4) Compute  $L = G [I - T] M G'$
- 5) Compute the two eigenvectors of  $L$ ,  $\lambda_1^j, \lambda_2^j$ , corresponding to its two smallest eigenvalues.
- 6) Compute the solution  $F = \lambda G [I - T]$

#### V. EXPERIMENTS

Due to the lack of space, we show here only one example of the method (Fig. 1). The original curve is the intersection of two cylinders with nonintersecting axes. Data points were generated in such a way that the distance to the curve has normal distribution, and not covering all the curve. The a priori assumption about the nature of the curve is that they are unrestricted quadric surfaces. The solution curve is not exactly equal to the original one, but it is very close in the region of space in which the data points lie.

#### VI. REPRESENTATION USING HIGHER DEGREE POLYNOMIALS

Due to the fact that this method allows us to work with higher degree polynomials, instead of representing an object as a collection of linear or quadratic patches we can represent the object as only one patch of higher degree. We even have the possibility of representing an object at different levels of resolution, using patches of different degrees. We have used this representation for position estimation [4]. We show here a planar example of an object composed of several linear or quadratic patches, which is approximated with only one polynomial patch of higher degree (Fig. 2).

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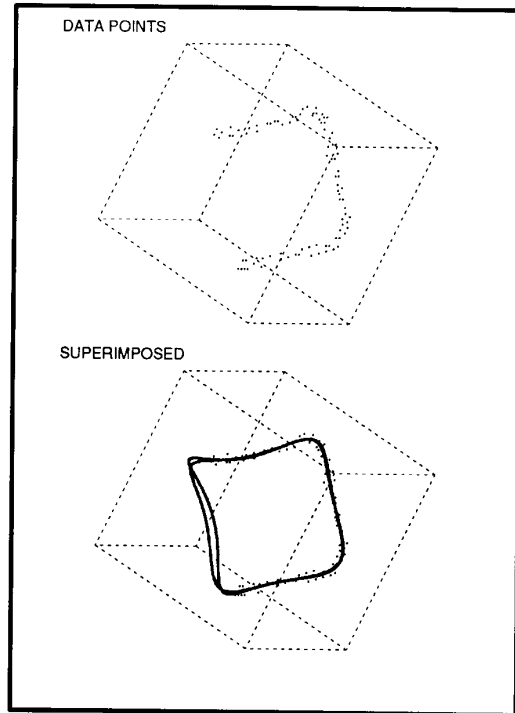


FIGURE 1 : Intersection of two cylinders

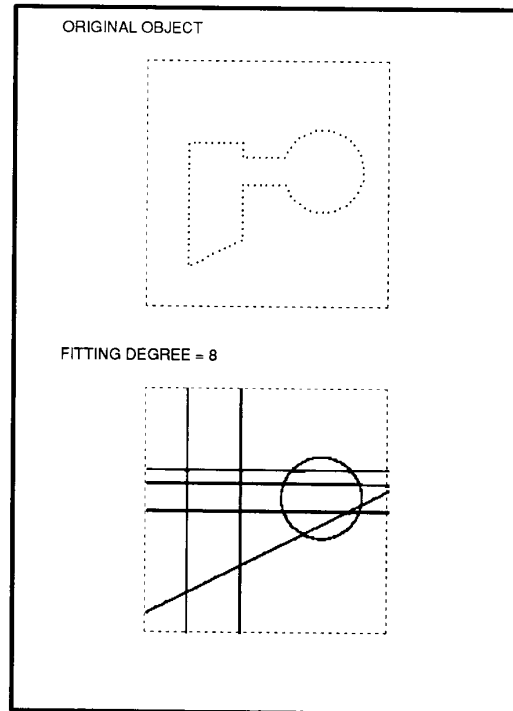


FIGURE 2 : The original data points are included in the set of roots of one polynomial of degree 8.