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Dual Mesh Resampling

Gabriel Taubin

IBM T.J. Watson Research Center *

Abstract

The dual of a 2-manifold polygonal mesh without boundary is commonly defined as another mesh with the same topology (genus) but different connectivity (vertex-face incidence), in which faces and vertices occupy complementary locations and the position of each dual vertex is computed as the center of mass (barycenter or centroid) of the vertices that support the corresponding face. This *barycenter dual* mesh operator is connectivity idempotent but not geometrically idempotent for any choice of vertex positions, other than constants. In this paper we construct a new *resampling dual* mesh operator that is geometrically idempotent for the largest possible linear subspace of vertex positions. We look at the primal and dual mesh connectivities as irregular sampling spaces, and at the rules to determine dual vertex positions as the result of a resampling process that minimizes signal loss. Our formulation, motivated by the duality of Platonic solids, requires the solution of a simple least-squares problem. We introduce a simple and efficient iterative algorithm closely related to Laplacian smoothing, and with the same computational cost. We also characterize the configurations of vertex positions where signal loss does and does not occur during dual mesh resampling, and the asymptotic behavior of iterative dual mesh resampling in the general case. Finally, we describe the close relation existing with discrete fairing and variational subdivision, and define a new *primal-dual* interpolatory recursive subdivision scheme.

CR Categories and Subject Descriptors:
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1 Introduction

A polygonal mesh is defined by the association between the faces and their sustaining vertices (connectivity), by the vertex positions (geometry), and by optional colors, normals and texture coordinates (properties). Properties can be bound to the vertices, faces, or corners of the mesh, but it is sufficient to consider meshes with vertex positions and no other properties. This is so because: properties bound per vertex can be treated in the same way as vertex positions, and properties bound per face or per corner can be regarded as bound per vertex to a closely related mesh (dual mesh in the per face case, Doo-Sabin [4] connectivity in the per corner case).

In general we look at vertex positions as signals defined on the mesh connectivity. In section 2 we review some basic concepts about meshes and mesh signals, and establish the notation for the rest of the paper. The result of applying the *barycenter dual* mesh operator to a manifold polygonal mesh without boundary (the *primal mesh*) is another mesh with the same topology but different

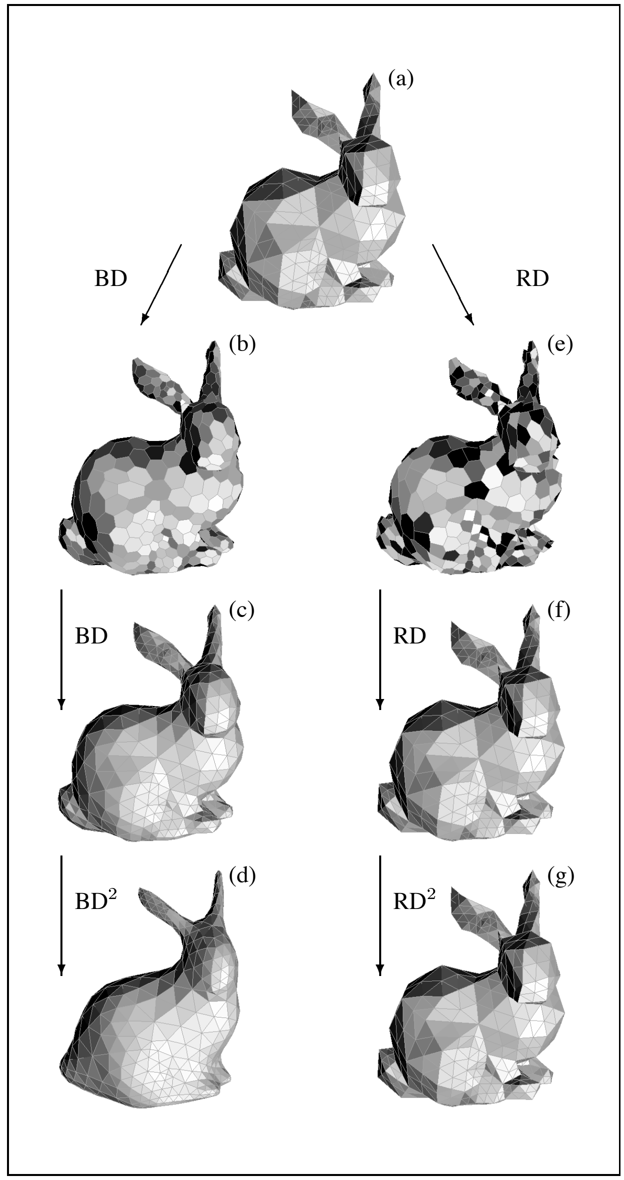
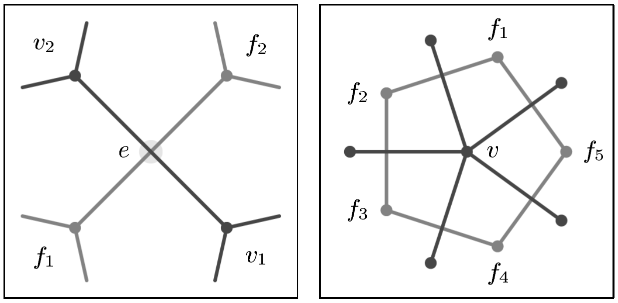


Figure 1: Barycenter dual mesh operator (BD) vs. resampling dual mesh operator (RD). (a): primal mesh; (b) dual mesh; (c) D^2 applied to primal mesh has same connectivity but different geometry; (d) D^4 applied to primal mesh displays evident shrinkage; (e): resampling dual mesh; (f): primal geometry is recovered when RD^2 is applied to primal mesh; (g): RD^4 applied to primal mesh also recovers primal geometry. In general, some signal loss may occur when RD^2 is applied to the primal mesh, but the sequence RD^{2n} converges fast.

*IBM T.J. Watson Research Center, Yorktown Heights, NY 10598 `taubin@us.ibm.com`. Work performed at the California Institute of Technology during the 2000-2001 academic year, while on sabbatical from IBM Research as Visiting Professor of Electrical Engineering



A B

Figure 2: A: an edge connects two vertices and two incident faces. B: a dual mesh face corresponds to cycle in the dual graph around a primal vertex.

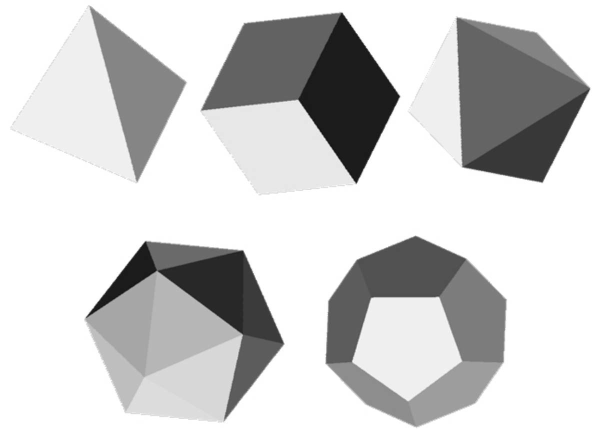


Figure 3: Platonic solids: tetrahedron, cube, octahedron, icosahedron, and dodecahedron.

connectivity (the *barycenter dual mesh*), in which faces and vertices occupy complementary locations, and the centers of mass (barycenters or centroids) of the vertices supporting the primal faces define the dual vertex positions. In section 3 we discuss this classical construction in more detail. For example, a primal mesh is shown in figure 1-(a), and the result of applying the dual mesh operator to it is shown in figure 1-(b). We claim that this operator is not properly named because the term *dual* is reserved in Mathematics to operators that are equal to the identity when squared (*idempotent operators*); and when the square of the barycenter dual mesh operator is applied to a mesh, the original connectivity is recovered but the vertex positions are not. In fact, the linear operator defined by the square of the barycenter dual mesh operator on the primal vertex positions is a second order smoothing operator that displays the same kind of shrinkage behavior as Laplacian smoothing [13], always producing shrinkage when applied to non-constant vertex positions. For example, figures 1-(c) and 1-(d) show the result of applying the square and the fourth power of the dual mesh operator to the primal mesh of figure 1-(a), respectively.

In this paper we look at the construction of the dual vertex positions as a resampling process, where the primal vertex positions, regarded as signals defined on the primal mesh connectivity, are linearly resampled (transferred) according to the dual mesh connectivity. The problem is how to define resampling rules *as a function of the connectivity* so that loss of information is minimized, i.e., so that in general the original signal is recovered when the same process is applied to the resampled signal (dual vertex positions) on the dual mesh connectivity. Note that this is not the case when the dual vertex positions are defined as the barycenters of the faces. Here the result of resampling back from the dual to the primal sampling space always produces loss of signal, unless the signal is constant (zero frequency).

In section 4 we show that the dual vertex positions of the Platonic solids [17] circumscribed by a common sphere can also be defined as the solution of a least-squares problem with a quadratic energy function linking primal and dual vertex positions, and that this energy function is well defined for any manifold polygonal mesh without boundary. In section 5 we derive explicit expressions for the new resampling dual vertex positions as linear functions of the primal vertex positions. In section 6 we rewrite the formula for the resampling dual vertex positions as a function of the primal and dual Laplacian operators, and we show that the linear operator defined by the square of the resampling dual mesh operator is a smoothing operator that prevents shrinkage as in Taubin's $\lambda|\mu$ smoothing algorithm [13]. The new expression for the resampling dual vertex positions with the Laplacian operator leads to an efficient algorithm, described in section 7, to compute the resampling dual vertex positions. Even though this algorithm, which can be implemented as a minor modification of the Laplacian smoothing algorithm, converges very fast, we exploit the relation to Laplacian smoothing even further, and define approximate algorithms that run in a fraction of the time in a pre-determined number of operations.

In the classical problem of uniform sampling rate conversion in signal processing [2, 16], under conditions determined by Shanon's sampling theorem, when the sampling rate is reduced (fewer faces than vertices in the primal mesh) the frequency content of the signal determines whether loss of information (due to aliasing) occurs or not, and when the sampling rate is increased (more faces than vertices in the primal mesh), no loss of information occurs because the resampled signal is of low frequency. The situation here is more complex, due to lack of regularity, but in section 8 we establish the conditions under which loss of information occurs and is prevented, and study the asymptotic behavior of iterative dual mesh resampling, in fact defining the space of low frequency signals, i.e. the largest linear subspace of signals that can be resampled with no loss of information. For example, figures 1-(e), 1-(f), and 1-(g), shows the result of applying the first, second, and fourth powers of our new *resampling dual mesh* operator to the primal mesh of figure 1-(a). Note that the second and fourth powers (and any even power) recover the primal mesh because the conditions for lossless dual mesh resampling are satisfied.

In addition to Taubin's low-pass filter algorithms [13, 15], a number of enhancements have been introduced in recent years to Laplacian smoothing to try to overcome some of its limitations, such as prevention of tangential drift [7, 3], implicit fairing for aggressive smoothing [3], the variational approach for interpolatory fairing [11], and the explicit incorporation of normals in the smoothing process for better control in shape design [18]. The algorithms introduced in this paper have potential applications in these areas. We do not explore these applications here, but we in section 10 we define a new interpolatory recursive subdivision scheme based on the *primal-dual mesh* operator, and we study the relation with variational fairing in section 12. Finally, we present our conclusions and plans for future work in section 13.

2 Meshes and Signals

The connectivity of a polygonal mesh M is defined by the incidence relationships existing among its V vertices, E edges, and F faces. We also use the symbols V, E, and F to denote the *sets* of vertices, edges, and faces of M. A *boundary* edge of a polygonal mesh has

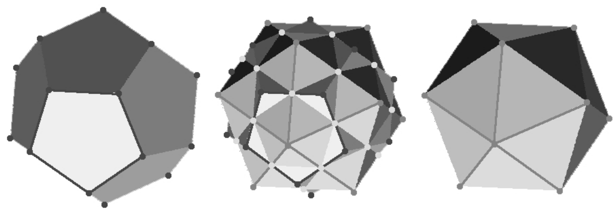


Figure 4: Platonic duals: icosahedron and dodecahedron.

exactly one incident face, a *regular* edge has two incident faces, and a *singular* edge has three or more incident faces. The *dual graph* of a polygonal mesh is the graph defined by the mesh faces as graph vertices, and the regular mesh edges as graph edges.

When we refer to a *mesh* in this paper, we mean a *2-manifold polygonal mesh without boundary*. No other types of polygonal meshes, such as meshes with boundary or non-manifold meshes, will be considered here. Extensions of the techniques introduced in this paper to meshes with boundary and non-manifold are possible, but will be done elsewhere.

So, our meshes have no isolated vertices, i.e., every vertex is the corner of at least one face. Although the methods described in this paper work for meshes with multiple connected components, it is sufficient to consider connected meshes, because all the operations can be decomposed into independent operations on the connected components. The concepts of orientation and orientability play no role in this paper, and will be ignored as well. In our meshes every edge is regular, and the subgraph of the dual graph defined by all the faces incident to each mesh vertex form a closed loop, or cycle of faces. Figure 2 illustrates these concepts. The connectivity of the *dual mesh* of M is defined by the primal faces as dual vertices, and these dual graph loops as dual faces. Since each primal edge connects two vertices and has two incident faces, we identify primal and dual edges, and refer to them as just edges. Figures 1-(a) and 1-(b) show a mesh, and its dual.

We consider vertex, edge, and face *signals* defined on the vertices, edges, and faces of a mesh, i.e., on the different connectivity elements. These signals define vector spaces. For example, primal vertex positions are three-dimensional vertex signals, and dual vertex positions are three-dimensional face signals (vertex signals on the dual mesh). The role of the edge signals will become evident in subsequent sections. Since all the computations in this paper are linear and can be performed on each vertex coordinate independently, it is sufficient to consider one-dimensional signals. We arrange these one-dimensional signals as column vectors X_V, X_E, and X_F, of dimension V, E, and F, respectively. The element of X_V corresponding to a vertex v is denoted x_v, the element X_E corresponding to an edge e is denoted x_e, and the element of X_F corresponding to a face f is denoted x_f.

3 The Barycenter Dual Mesh

The *quad-edge* data structure [6] can be used to efficiently represent and traverse a mesh, and in particular to construct the connectivity of the dual mesh. The faces of the dual mesh can be reconstructed by cycling around each vertex of the primal mesh using the information stored in the quad-edge data structure. In the dual mesh construction the dual vertex signal corresponding to a face $f = (v_1, \ldots, v_n)$ with n corners is computed as the average of the primal vertex signals corresponding to the corners of the face

$$x_f = \frac{1}{n}\sum_{i=1}^{n} x_{v_i} \ .$$

We can also write this assignment in vector form as

$$X_F = W_{FV} X_V \ , \tag{1}$$

where W_{FV} is the vertex-face incident matrix I_{FV} normalized so that the sum of each row is equal to one. If this construction is repeated on the dual mesh, we obtain a mesh with the same connectivity as the primal mesh, but with vertex positions

$$X'_V = W_{VF} W_{FV} X_V \ ,$$

where the matrix W_{VF} is the face-vertex incident matrix I_{VF} normalized so that the sum of each row is equal to one. The matrix $W_{VF}W_{FV}$ is not symmetric, but is composed of non-negative elements, and its rows add up to one. It defines a second order smoothing operator closely related to Laplacian smoothing [13].

If we look at the set of vertex signals such that $X'_V = X_V$, i.e., the invariant subspace of $W_{VF}W_{FV}$ associated with the eigenvalue 1, we generally end up with a subspace spanned by the constant vector $X_V = (1, \ldots, 1)^t$. Our approach,described in the next three sections, is to construct new matrices W_{FEV} and W_{VEF}, *as functions of the connectivity*, to replace the matrix W_{FV} in the construction of dual vertex signal values, in such a way that the dimension of the invariant subspace of $W_{VEF}W_{FEV}$ associated with the eigenvalue 1 is maximized.

4 Platonic Solids

Figure 3 shows the five Platonic solids: the tetrahedron, the cube, the octahedron, the icosahedron, and the dodecahedron. All of them are circumscribed by a sphere, say of unit radius. In terms of connectivity, the tetrahedron is dual of itself, and both the cube and the octahedron, and the icosahedron and the dodecahedron, are dual of each other. Because of the symmetries, if we construct the dual mesh of each of these meshes as described in section 3, with the dual vertex positions at the barycenters of the primal faces, we end up with the corresponding dual platonic solids, but circumscribed by spheres of smaller radii. This can be solved by adjusting the scale, moving the face positions away from the center of the primal mesh along the corresponding radial directions until the dual vertex positions are circumscribed by the unit sphere. This procedure solves the problem for the Platonic solids, but it does not work for other more general meshes. However, the construction has the following property [17], that can be observed in figure 4 for the case of the icosahedron and the dodecahedron: for each edge $e = \{v_1, v_2, f_1, f_2\}$ connecting two vertices and two faces, the segments joining the corresponding vertex positions and face positions intersect at their midpoints, i.e.,

$$\frac{1}{2}(x_{v_1} + x_{v_2}) = \frac{1}{2}(x_{f_1} + x_{f_2}) \ .$$

This means that the construction of the dual vertex positions of the Platonic solids can be described as the minimization of the following energy function

$$\phi(X_V, X_F) = \sum_{e \in E} \|x_{v_1} + x_{v_2} - x_{f_1} - x_{f_2}\|^2 \tag{2}$$

with respect to X_F with X_V fixed, the sum taken over all the edges of the mesh. The value attained at the minimum is zero. Note that the vertex positions of the double dual mesh are obtained by minimizing the same energy function with respect to X_V with X_F fixed. In addition, this energy function is defined for every manifold polygonal mesh without boundary.

`PrimalLaplacian`(X_V)
  # accumulate
  $Y_V = 0$;
  `for` $e = (v_1, v_2, f_1, f_2) \in E$
    $y_{v_1} = y_{v_1} + (x_{v_1} - x_{v_2})$;
    $y_{v_2} = y_{v_2} + (x_{v_2} - x_{v_1})$;
  `end`;
  # normalize
  `for` $v \in V$
    $y_v = y_v / |v^\star|$;
  `end`;
  # return $K_V X_V$
`return` Y_V;

`DualLaplacian`(X_F)
  # accumulate
  $Y_F = 0$;
  `for` $e = (v_1, v_2, f_1, f_2) \in E$
    $y_{f_1} = y_{f_1} + (x_{f_1} - x_{f_2})$;
    $y_{f_2} = y_{f_2} + (x_{f_2} - x_{f_1})$;
  `end`;
  # normalize
  `for` $f \in F$
    $y_f = y_f / |f^\star|$;
  `end`;
  # return $K_F X_F$
`return` Y_F;

Figure 5: Algorithms to evaluate the primal ($K_V X_V$) and dual ($K_F X_F$) Laplacian operators by traversing the mesh edges

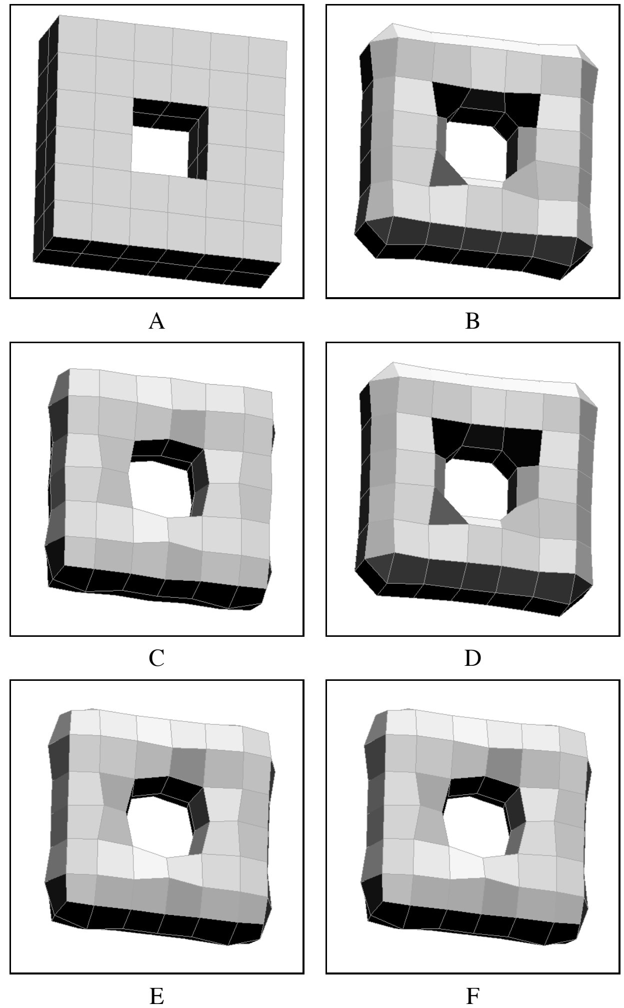


Figure 6: Resampling with rank-deficient $I^t_{EF} I_{EV}$. (A) primal mesh (B) resampling dual mesh, (C) second power of resampling dual mesh, (D) third power, (E) fourth power (P^2), and (F) sixth power (P^3).

5 The Resampling Dual Mesh

Equation 2 can be written in matrix form for any mesh as follows

$$\phi(X_V, X_F) = \|I_{EV} X_V - I_{EF} X_F\|^2 \tag{3}$$

where I_{EV} and I_{EF} are the edge-vertex and edge-face incidence matrices without normalization. In general, these are full-rank matrices. Since equation 3 is quadratic, the minimizer of $\phi(X_V, \star)$ is the solution of the linear system

$$I^t_{EF} I_{EF} X_F = I^t_{EF} I_{EV} X_V \ , \tag{4}$$

which is obtained by differentiating ϕ with respect to X_F, or equivalently

$$X'_F = W_{FEV} X_V$$

with

$$W_{FEV} = (I^t_{EF} I_{EF})^{-1} I^t_{EF} I_{EV} \ . \tag{5}$$

And due to symmetry, the minimizer of $\phi(\star, X_F)$ can be written as

$$X'_V = W_{VEF} X_F \ ,$$

with

$$W_{VEF} = (I^t_{EV} I_{EV})^{-1} I^t_{EV} I_{EF} \ . \tag{6}$$

The matrix in equations 5 can also be written as

$$W_{FEV} = I^\dagger_{EF} I_{EV} \ ,$$

where

$$I^\dagger_{EV} = (I^t_{EV} I_{EV})^{-1} I^t_{EV}$$

is the *pseudo-inverse* of the matrix I_{EV}. A similar expression can be written for the matrix W_{VEF} of equation 6.

6 Relation to Laplacian Smoothing

In this section we establish the relation between dual resampling formula $X'_V = W_{VEF} X_F$ and Laplacian smoothing. In section 7 we use this formulation to define a simple and efficient algorithm to evaluate the resampling dual vertex signals as a minor modification of the Laplacian smoothing algorithm.

In its simplest form, the primal Laplacian operator is defined for a vector of vertex positions X_V as

$$\Delta_V x_v = \sum_{v' \in v^\star} w_{vv'} (x_v - x_{v'})$$

where $v^\star$ is the set of vertices v' connected to vertex v by an edge, the weight $w_{vv'}$ is equal to $1/|v^\star|$, and $|v^\star|$ is the number of elements in the set $v^\star$. If we organize the weights as a matrix W_V, we can write the Laplacian operator in matrix form as follows

$$\Delta_V X_V = -K_V X_V \ ,$$

where $K_V = I_V - W_V$ has eigenvalues in $[0, 2]$, and I_V is the identity matrix in the space of vertex signals [13]. A similar expression can be written for the dual Laplacian operator $\Delta_F X_F$. Pseudocode implementations of the algorithms to evaluate the primal and dual laplacian operators are described in figure 3.

Note that the diagonal element of the matrix $I^t_{EV} I_{EV}$ corresponding to a vertex v is equal to the number $|v^\star|$ of vertices connected to v through an edge, and if we organize these numbers as a diagonal matrix D_V, we have

$$I^t_{EV} I_{EV} = D_V(I_V + W_V) = 2D_V(I_V - \lambda K_V) \ ,$$

with $\lambda = 0.5$, and also

$$I^t_{EV} I_{EF} = 2D_V W_{VF} \ ,$$

where W_{VF} is the matrix introduced in section 3 (I_{VF} normalized so that the sum of each row is equal to one). This allows us to rewrite the equation (dual of 4) used to compute the double dual vertex positions as a function of the face positions, as follows

$$(I_V - \lambda K_V) X'_V = W_{VF} X_F \ . \qquad (7)$$

With a similar derivation, we can rewrite equation 4, used to compute the face positions as a function of the primal vertex positions, as follows

$$(I_F - \lambda K_F) X'_F = W_{FV} X_V \ , \qquad (8)$$

where K_F is the matrix of the Laplacian operator defined on the dual mesh.

Note that in equation 8, the face positions are computed by applying implicit smoothing [3] to the barycenters of the faces with *negative* time step $dt = -0.5$. This process is not a smoother, but actually *enhances* high frequencies. The behavior of this process is closely related to Taubin's $\lambda|\mu$ non-shrinking smoothing algorithm [13], where a true low pass-filter is constructed by two steps of Laplacian smoothing with positive (high frequency attenuating) and negative (high frequency enhancing) scaling factors. Here the computation of face barycenters has a high frequency attenuating effect, and the implicit smoother with negative time step has a high frequency enhancing effect:

$$W_{FEV} = (I_F - \lambda K_F)^{-1} W_{FV} \ .$$

The final result, as in Taubin's $\lambda|\mu$ algorithm, is a low-pass filter effect without shrinkage, while the data is transferred from the primal to the dual mesh.

7 Algorithm

To compute the dual vertex signals $X'_F = W_{FEV} X_V$ as a function of the primal vertex signals we solve the linear system of equation 8 using a simple iterative method, which, as we will see in this section, is a minor modification of the Laplacian smoothing algorithm.

Iterative methods are used to solve systems of linear equations such as

$$AY = Z \ , \qquad (9)$$

where the non-singular square matrix A is large and sparse, and Y and Z are vectors of the same dimension [5]. Several popular iterative solvers, such as Jacobi and Gauss-Seidel, are based on the following general structure. By decomposing the matrix A as the sum of two square matrices $A = B + C$, such that B is easy to invert and the spectral radius of the matrix $H = -B^{-1}C$ is less than one, the problem is reduced to the solution of the equivalent system

$$(I - H)\, Y = Y_0 \ ,$$

```
PrimalDualSmoothing (X_V, n, λ, steps)
  for s = 0,...,steps − 1
    X_{F,0} = W_{FV} X_V;
    for j = 0,...,n − 1
      dX_F = DualLaplacian (X_{F,j});
      X_{F,j+1} = X_{F,0} + λ dX_F;
    end;
    X_{V,0} = W_{VF} X_{F,n};
    for j = 0,...,n − 1
      dX_V = PrimalLaplacian (X_{V,j});
      X_{V,j+1} = X_{V,0} + λ dX_V;
    end;
    X_V = X_{V,n};
  end;
return X_V;
```

Figure 7: Primal-dual smoothing algorithm. Pseudocode for the primal and dual Laplacian operators is described in figure 3.

with $Y_0 = B^{-1}Z$. The following simple algorithm

$$Y_n = Y_0 + H\, Y_{n-1} \qquad (10)$$

defines a sequence of estimates $\{Y_n : n = 1, 2, \ldots\}$ that converges to the solution of the original system of equations 9, because the series

$$\sum_{j=0}^{\infty} \theta^j = (I - \theta)^{-1}$$

converges absolutely and uniformly for $|\theta| < 1$, and

$$Y_n = \sum_{j=0}^{n} H^j\, Y_0 \ .$$

The rate of convergence is determined by the spectral radius ρ of H: if $\|H\, Y\| < \rho \|Y\|$ for all Y, and $0 \le \rho < 1$, then

$$\|Y - Y_n\| \le \sum_{j=n+1}^{\infty} \|H^j\, Y_0\| \le \sum_{j=n+1}^{\infty} \rho^j \|Y_0\| = \frac{\rho^{n+1}}{1-\rho} \|Y_0\| \ .$$

For example, if $\rho \le 1/2$, the relative error is less than 0.1% after ten iterations, and the estimates have about six correct digits after twenty iterations.

To solve equation 8 we set $H = \lambda K_F$, $Y = X_F$, and $Y_0 = W_{FV} X_V$. Although the spectral radius H is bound above by 1 (because the eigenvalues are in the interval $[0, 1]$), in typical meshes this upper bound is closer to $1/2$, and we observe in practice convergence to an error of less than 0.1% after ten iterations.

Note that, if we replace Y_0 by Y_{n-1} in the iteration rule described in equation 10 we obtain

$$Y_n = Y_{n-1} + H\, Y_{n-1} \ ,$$

or equivalently

$$Y_n = (I + H)^n Y_0 \ ,$$

which corresponds to n steps of the Laplacian smoothing with parameter $\lambda = 1/2$. The main difference is that in Laplacian smoothing the number of iterations is specified in advance, while in our new algorithm it depends on an error criterion. As an alternative, we can use the new algorithm with both an error tolerance and a maximum number of iterations, and stop as soon as either stopping criterion is satisfied. In our experience, a maximum number of iterations of 20 and error tolerance of 0.001 produces excellent results.

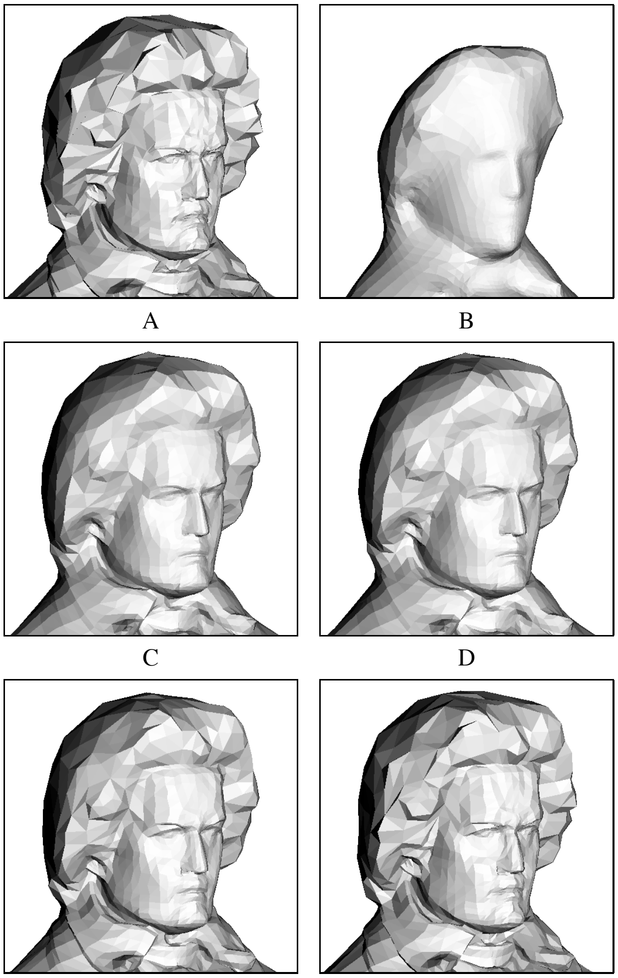


Figure 8: Primal-Dual smoothing vs. Laplacian smoothing. (A) a mesh. The result of applying (B) 12 Laplacian smoothing steps with parameter $\lambda = 0.6307$, and (C) 12 steps of Taubin's smoothing algorithm with parameters $\lambda = 0.6307$ and $\mu = -0.6732$. The result of applying primal-dual smoothing steps with parameters $\lambda = 0.5$: (D) 6 steps with $n = 1$, (E) 3 steps with $n = 2$, and (F) 1 step with $n = 6$. The computational cost is about the same in all cases.

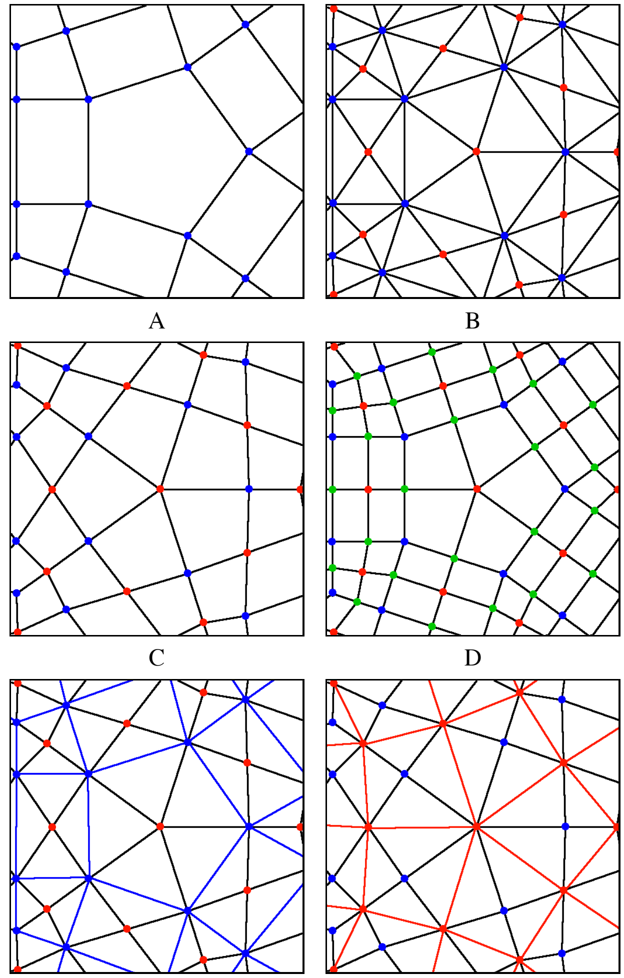


Figure 9: The Primal-Dual mesh. (A) a coarse mesh, (B) the faces are triangulated by connecting the new face vertices (red) to the original vertices (blue), (C) the primal-dual connectivity is obtained by removing the original edges, (D) the Catmull-Clark connectivity is obtained after a second primal-dual refinement step. (E) the primal connectivity can be recovered from the primal-dual connectivity by inserting the primal diagonals and removing the dual vertices. (F) the dual connectivity can be recovered by inserting the dual diagonals and removing the primal vertices.

8 Analysis of Dual Mesh Resampling

In this section, and based on simple concepts from Linear Algebra, we establish necessary and sufficient conditions under which no loss of information occurs when primal vertex signals are resampled, and describe the general behavior of the dual resampling process.

The matrix I_{EV} defines a linear mapping from the space of vertex signals into the space of edge signals $X_E = I_{EV}X_V$. Since normally meshes have more edges than vertices and the matrix I_{EV} is full-rank, the image of this mapping is a subspace S_V of dimensions V in the space of edge signals. Let T_V be the orthogonal complement of S_V in the space of edge signals, i.e., $S_V \oplus T_V$ is the full space of edge signals. Every edge signal can be decomposed in a unique way as a sum of two edge signals; a first one $P_V X_E$ belonging to S_V and a second one $(I_E - P_V)X_E$ belonging to T_V

$$X_E = P_V X_E + (I_E - P_V)X_E \ ,$$

where $P_V = I_{EV}I_{EV}^{\dagger}$, and I_E is the identity in the space of edge signals. The pseudo-inverse $I_{EV}^{\dagger}$ of the matrix I_{EV} defines a linear mapping from the space of edge signals into the space of vertex signals $X'_V = I_{EV}^{\dagger}X_E$ that recovers the vertex signal part of any edge signal

$$X_V = I_{EV}^{\dagger}I_{EV}X_V$$

because $I_{EV}^{\dagger}I_{EV} = I_V$ (with I_V the identity of vertex signals). The matrix P_V is a projector ($P_V^2 = P_V$) in the space of edge signals which has S_V as its invariant subspace associated with the eigenvalue 1 and T_V as its invariant subspace associated with the eigenvalue 0. The matrix I_{EF} also defines a linear mapping from the space of face signals into the space of edges signals $X_E = I_{EF}X_F$, orthogonal subspaces S_F and T_F of edge signals, and a projector $P_F = I_{EF}I_{EF}^{\dagger}$.

The intersection $S = S_V \cap S_F$ of the subspaces S_V and S_F

plays a key role in determining whether signal loss occurs or not in the dual mesh resampling process. We call a vertex signal X_V *dual resamplable* if $PX_V = X_V$, i.e., if the dual resampling process produces no loss of information. These signals correspond to edge signals $X_E = I_{EV}X_V$ that belong to S. To prove this statement, let $P = W_{VEF}W_{FEV}$ be the matrix corresponding to the square of the dual mesh resampling process, and let X_V be a vertex signal. If the corresponding edge signal $X_E = I_{EV}X_V$ belongs to S, then there is a face signal X_F so that $X_E = I_{EF}X_F$. It follows that

$$\begin{aligned} PX_V &= I_{EV}^{\dagger}I_{EF}I_{EF}^{\dagger}I_{EV}X_V \\ &= I_{EV}^{\dagger}I_{EF}I_{EF}^{\dagger}I_{EF}X_F \\ &= I_{EV}^{\dagger}I_{EF}X_F \\ &= I_{EV}^{\dagger}I_{EV}X_V \\ &= X_V \end{aligned}$$

Note that since the subspaces S_V and S_F are spanned by the columns of the matrices I_{EV} and I_{EF}, the dimension of S is equal to the rank of the matrix $I_{EF}^t I_{EV}$. We have three particular cases: 1) the dimension of S is V (S_V is a subspace of S_F); 2) the dimension of S is F (S_F is a subspace of S_V); and 3) the dimension of S is strictly less than the minimum of V and F. In the first case (sampling rate increase), no loss of information occurs for any vertex signal, i.e., $P = I_V$. In the second case (sampling rate decrease) the process is idempotent, i.e., $P^2 = P$. This is so because since $S_F \subseteq S_V$ we have $P_V P_F = P_F$, and so

$$\begin{aligned} P^2 &= I_{EV}^{\dagger}P_F P_V P_F I_{EV}X_V &&= I_{EV}^{\dagger}P_F^2 I_{EV}X_V \\ &= I_{EV}^{\dagger}P_F I_{EV}X_V &&= P\ . \end{aligned}$$

Neither one of these first two cases are very common. Most typically we encounter the third case, in which iterative dual mesh resampling produces a sequence of vertex signals that quickly converges to a resamplable one:

Proposition 1 *(lossy resampling) For any vertex signal X_V the sequence $P^n X_V$ converges to a dual resamplable signal.*

Proof 1 : Let us define the following sequence of vertex signals

$$\begin{cases} X_V^0 &= X_V \\ X_V^n &= P\,X_V^{n-1} \quad n > 0 \end{cases}$$

Clearly, if the sequence converges, the limit vector X_V^{∞} satisfies the desired property $PX_V^{\infty} = X_V^{\infty}$. To show convergence, it is sufficient to prove that the sequence $I_{EV}X_V^n$ converges. Our argument is based on an eigenvalue analysis. Note that

$$I_{EV}X_V^n = I_{EV}P^n X_V = (P_V P_F)^n I_{EV}X_V\ ,$$

and since P_V is a projector, and $P_V I_{EV}X_V = I_{EV}X_V$, we have

$$(P_V P_F)^n I_{EV}X_V = (P_V P_F P_V)^n I_{EV}X_V\ .$$

Now, the matrix $P_V P_F P_V$ is symmetric and non-negative definite. Since P_V and P_F are projectors, the eigenvalues of $P_V P_F P_V$ are between zero and one, with eigenvalue 1 corresponding to eigenvectors in S, and eigenvalues strictly less than 1 corresponding to eigenvectors orthogonal to S. Let λ be the largest of the eigenvalues less than 1, which is equal to the cosine of the angle between the subspaces $S_V \ominus S$ and $S_F \ominus S$. While the projection of $I_{EV}X_V^n$ onto S stays constant, the projection onto the orthogonal subspace to S converges to zero at least as fast as λ^n. $\square$

Figure 6 illustrates this last case. Numerical algorithms to determine the rank of the matrix $I_{EF}^t I_{EV}$ can be based on the QR decomposition for small meshes, and the SVD algorithm for large meshes [5]. Computing the smallest singular value would be sufficient to know whether we are in cases 1 or 2, or 3. Further work is needed to relate local combinatorial relations between vertices, faces, and edges to the rank of the matrix $I_{EF}^t I_{EV}$.

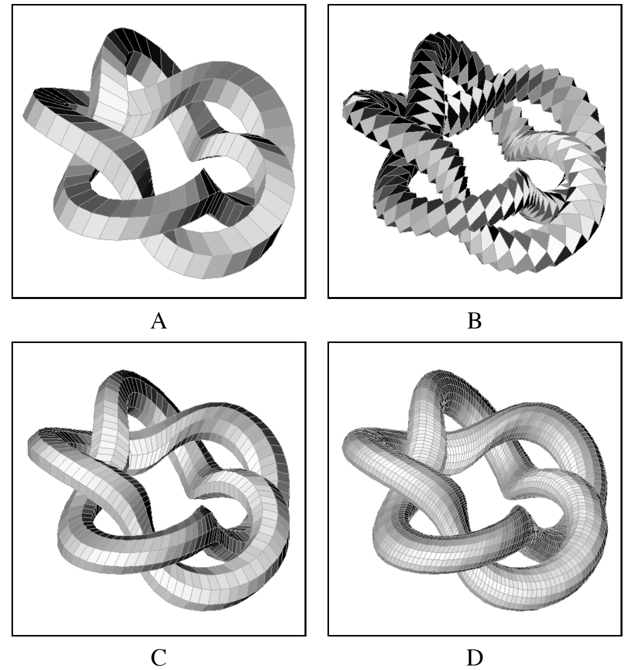


Figure 10: The primal-dual mesh operator applied recursively to a coarse mesh (A), once (B), twice (C), and four times (D). These subdivision meshes are fair in the variational sense.

9 Primal-Dual Smoothing

If we use the new algorithm to compute the dual vertex positions by specifying just a maximum number of iterations, and then apply the same algorithm to recompute primal vertex positions as a function of the dual vertex positions, we obtain a new family of non-shrinking smoothing operators for the primal vertex signals $X'_V = P_n X_V$ described by the following steps

$$\begin{aligned} &1)\quad X'_F &&= (I_F + \cdots + \lambda^n K_F^n)\,W_{FV}X_V \\ &2)\quad X'_V &&= (I_V + \cdots + \lambda^n K_V^n)\,W_{VF}X'_F \end{aligned}$$

and illustrated in pseudocode in figure 7.

Since for large n we have $X'_V \approx PX_V$, with $P = W_{VEF}W_{FEV}$ which satisfies $P^2 \approx P$, as n increases, these operators produce *less* smoothing. We also have the freedom of playing with the parameter λ. Figure 8 shows some results compared to Laplacian and Taubin's smoothing algorithms.

Note that to implement the *primal-dual smoothing algorithm* we do not need to construct the connectivity of the dual mesh explicitly. The two steps are based on recursively evaluating products of the Laplacian matrices K_V and K_F by vectors of dimensions V and F, and by accumulating partial results in temporary arrays of the same dimensions. But both matrix vector products can be accumulated by traversing the same list of mesh edges.

10 The Primal-Dual Mesh

As noted by Kobbelt [10], the operator that transforms the connectivity of a mesh into its Catmull-Clark connectivity [1] has a square root. The result of applying this square root operator to the connectivity of a mesh has the vertices and faces of the original mesh as vertices, the edges of the original face as quadrilateral faces, and the vertex-face incident pairs as edges. The *quad-edge* data structure [6] can be used to operate on the primal-dual mesh. Figure 9

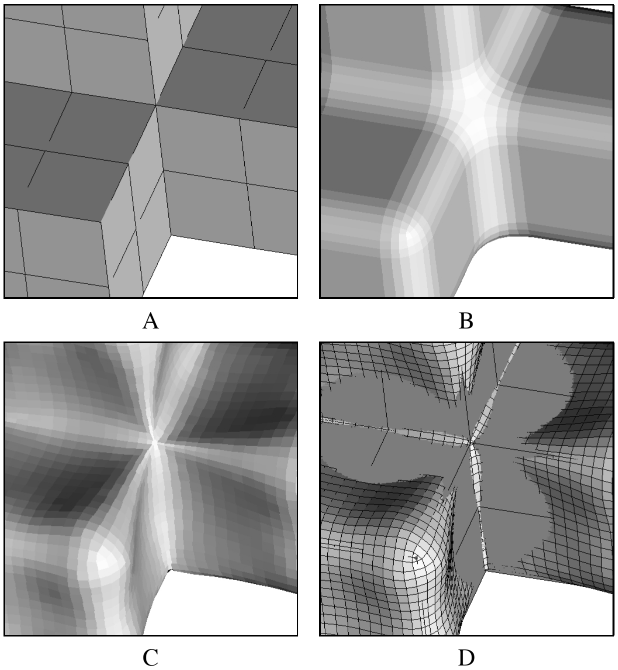


Figure 11: Non-shrinking Doo-Sabin subdivision operator is the composition of the square of the primal-dual operator followed by the resampling dual operator. (A) coarse mesh, (B) coarse mesh after three shrinking Doo-Sabin refinement steps, (C) coarse mesh after three non-shrinking Doo-Sabin refinement steps, (D) superposition of (A) and (C).

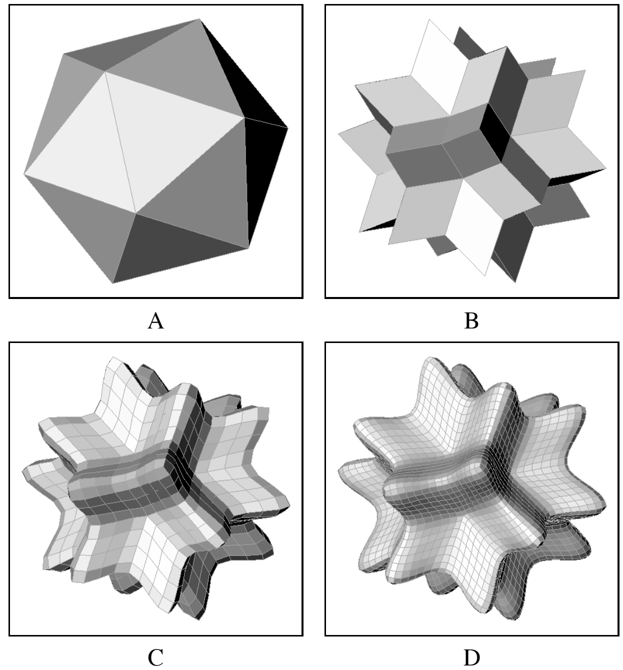


Figure 12: Surface designed by combining resampling-dual (RD) and primal-dual (PD) mesh operators. (A) coarse mesh, the result of applying (B) $PD \circ RD \circ PD$, (C) $PD^5 \circ RD \circ PD$, and (D) $PD^7 \circ RD \circ PD$ to the coarse mesh.

illustrates the construction, and how to recover the connectivity of the original mesh and its dual from the resulting mesh connectivity.

If we add to this connectivity refinement operator our algorithm to compute the resampling dual vertex positions, we obtain an interpolatory refinement mesh operator. Because of the symmetric role that primal and dual vertices play in this construction, we prefer to call it the *primal-dual mesh* operator. Note that this operator has two inverses that can be used to recover either the original mesh, or the resampling dual mesh. Also, since the scheme is interpolatory, and the original vertices are a subset of the vertices of the resulting mesh, there is no loss of information.

The primal-dual mesh operator defines a linear operator that maps vertex signals on the primal mesh to vertex signals on the primal-dual mesh

$$X_V \mapsto \begin{pmatrix} X_V \\ X_F \end{pmatrix} .$$

Since the matrix that defines this linear operator is full-rank, the image is a subspace of dimension V. And this is true even if the primal-dual mesh operator is applied iteratively several times to refine the mesh more and more. We will see in section 12 that these meshes are smooth in the variational sense. Figure 10 shows an example of applying the primal-dual mesh operator recursively several times to a coarse mesh as a mesh design tool.

11 Non-Shrinking Doo-Sabin

Since the Doo-Sabin connectivity of a mesh is the dual of the Catmull-Clark connectivity, and this is the square of the primal-dual connectivity, we can combine the resampling dual and primal-dual mesh operators to produce a non-shrinking version of the Doo-Sabin [4] subdivision scheme: apply primal-dual twice followed by resampling dual. An example is shown in figure 11. This Doo-Sabin resampling operator is also an example of a resampling process to a different mesh with the same topology. More general cases will be studied in a subsequent paper.

Another application for other combinations of these two operators is as a design tool in an interactive modeling environment. One example of a surface designed in this way is shown in figure 12.

12 Relation to Variational Fairing

In this section we discuss the close relation existing between our primal-dual operator and the discrete fairing approach, which shows that the surfaces produced by recursive primal-dual subdivision are smooth in the variational sense. Further work is required to understand the local asymptotic behavior of primal-dual subdivision.

First of all, we modify the energy function $\phi(X_V, X_F)$ of equation 3 by introducing a symmetric positive definite $E \times E$ matrix H as follows

$$\phi_H(X_V, X_F) = X_E^t H X_E \ , \tag{11}$$

where X_E is the edge signal vector

$$X_E = I_{EV} X_V - I_{EF} X_F \ .$$

The qualitative behavior of the primal-dual subdivision process for different values of diagonally dominating H is very similar.

The Laplacian operator on the primal-dual mesh can be written in block matrix form as follows

$$\Delta \begin{pmatrix} X_V \\ X_F \end{pmatrix} = \begin{pmatrix} I_V & -W_{VF} \\ -W_{FV} & I_F \end{pmatrix} \begin{pmatrix} X_V \\ X_F \end{pmatrix}$$

where I_V and I_F are the identity matrices in the spaces of vertex and faces signals, respectively, and the matrices W_{VF} and W_{FV}

are the face-vertex and vertex-face incident matrices normalized so that each row adds up to one introduced in section 3.

In the Discrete Fairing approach [11], the smoothness of a mesh is increased by minimizing an energy function such as the square of the Laplacian

$$\left|\Delta \begin{pmatrix} X_V \\ X_F \end{pmatrix}\right|^2$$

with some vertices fixed, or other linear constraints. In our case we would minimize this expression with the primal vertex positions X_V fixed to obtain the dual vertex positions X_F, or with the dual vertex positions fixed to obtain the primal vertex positions.

If we replace the matrices W_{VF} and W_{FV} by the new matrices W_{VEF} and W_{FEV} defined in section 5, we obtain a new *resampling Laplacian* operator Δ_T which behaves in a very similar way

$$\Delta_T \begin{pmatrix} X_V \\ X_F \end{pmatrix} = \begin{pmatrix} I_V & -W_{VEF} \\ -W_{FEV} & I_F \end{pmatrix} \begin{pmatrix} X_V \\ X_F \end{pmatrix} .$$

But if we expand the square of the resampling Laplacian operator we obtain

$$|X_V - W_{VEF} X_F|^2 + |X_F - W_{FEV} X_V|^2 ,$$

or equivalently

$$\left|\Delta_T \begin{pmatrix} X_V \\ X_F \end{pmatrix}\right|^2 = |I_{EV}^\dagger X_E|^2 + |I_{EF}^\dagger X_E|^2 = \phi_H(X_V, X_F) ,$$

with

$$H = (I_{EV}^\dagger)^t I_{EV}^\dagger + (I_{EF}^\dagger)^t I_{EF}^\dagger .$$

13 Conclusions and Future Work

In this paper we described a solution to the problem of shrinkage in the construction of the dual mesh, introduced efficient algorithms to solve the problem, and shown some applications. Through a signal processing resampling point of view, we established necessary and sufficient conditions under which no loss of information occurs, and analyzed the asymptotic behavior of iterative dual mesh resampling.

We regard the results introduced in this paper as a first step toward a general theory for general mesh resampling, and complementary to existing approaches to remeshing [12], recursive subdivision [10, 20, 19], and 3D geometry compression [9, 8, 14]. We plan to explore these applications in subsequent papers.

In this paper we restricted our meshes to oriented manifold meshes without boundary. We also plan to extend the formulation and algorithms to meshes with boundary and non-singular edges. In this extended formulation we will have explicit parameters (boundary conditions), such as normals, associated with boundary and singular edges, that could be used very effectively in an interactive free-form shape design environment.

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